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1 Notation

1.1 Robot parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Parameter description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>141 gm</td>
<td>Mass of body.</td>
</tr>
<tr>
<td>$m_2$</td>
<td>16 gm</td>
<td>Mass of rear leg.</td>
</tr>
<tr>
<td>$I_1$</td>
<td>1300 gm – cm$^2$</td>
<td>Inertia of body.</td>
</tr>
<tr>
<td>$I_2$</td>
<td>30 gm – cm$^2$</td>
<td>Inertia of rear leg.</td>
</tr>
<tr>
<td>$r$</td>
<td>5.5 cm</td>
<td>Leg length.</td>
</tr>
<tr>
<td>$c_1$</td>
<td>1.76 cm</td>
<td>Distance of body COM along leg axis (positive is downward).</td>
</tr>
<tr>
<td>$w_1$</td>
<td>$0.75 \times 10^{-3}$ cm</td>
<td>Distance of body COM normal to leg axis (positive is backward).</td>
</tr>
<tr>
<td>$c_2$</td>
<td>3.85 cm</td>
<td>Distance of rear leg COM along leg axis (positive is downward).</td>
</tr>
<tr>
<td>$w_2$</td>
<td>$0.71 \times 10^{-3}$ cm</td>
<td>Distance of rear leg COM normal to leg axis (positive is backward).</td>
</tr>
<tr>
<td>$g$</td>
<td>9.81 m/s$^2$</td>
<td>Gravitational constant.</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.1372 rad</td>
<td>Ramp slope.</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.2182 rad</td>
<td>Maximum angle between the legs.</td>
</tr>
<tr>
<td>$C_1$</td>
<td>0.005 Ns/m</td>
<td>Viscous friction between rear leg and ground.</td>
</tr>
</tbody>
</table>

1.2 Simulation variables

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Variable description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>time.</td>
</tr>
<tr>
<td>$q_1$</td>
<td>absolute angle between the stance leg and normal to the ramp.</td>
</tr>
<tr>
<td>$q_2$</td>
<td>relative angle between the swing leg and the stance leg.</td>
</tr>
</tbody>
</table>
angular velocity of the stance leg (\(= \dot{\theta}_1\)).

\(u_2\) angular velocity of the swing leg (\(= \dot{\phi}_2\)).

\(q_{1}^+, q_{1}^-\) absolute angle of the stance leg wrt. vertical after and before transition.

\(u_1^+, u_1^-\) angular velocity of the stance leg wrt. vertical after and before transition.

\(q_{2}^+, q_{2}^-\) relative angle of the swing leg wrt. stance leg after and before transition.

\(u_2^+, u_2^-\) angular velocity of the swing leg wrt. stance leg after and before transition.

\(\vec{H}_P\) Angular momentum about point \(P\) after and before transition.

\(\dot{\vec{H}}_P\) Rate of change of angular momentum about point \(P\) and so on.

\(\vec{M}_P\) External angular moment about point \(P\) and so on.

## 2 Toy Design and 3D printing

![Figure 1: (A) UTSA Mascot “Rowdy” The Roadrunner Logo [2], (B) a sketch of the toy, and (C) a CAD rendering of the toy](image)

The Fig. 1(A) shows the UTSA mascot from the University’s Communications and Marketing website. The goal of the design stage was to add legs to the mascot, followed by tuning the mass distribution, leg geometry, and the tolerances on the hinge joint to create a functional walking prototype. The Fig. 1(B) shows the initial sketch of the toy with legs and Fig. 1(C) shows a 3D CAD rendering of the sketch that was drawn using SOLIDWORKS.

Next, the toy was pre-processed using CURA, an open source slicing software. CURA divides the CAD drawing into a number of slices as shown in Fig. 2. We also needed to specify other 3D printing options at this stage. The specifications included the: scaling, temperature of the nozzle, material fill for the toy and/or the support material. Note that we can only choose the in-fill for one of the two, toy or the support material and the other in-fill is automatically determined by CURA. These specifications are given in Tab. 1. CURA estimated the print time to be 15 hours.

Next the toy was printed using an Ultimaker 2, a desktop hobby-grade 3D printer. We used a 2.85mm diameter Polyactic Acid (PLA) filament for 3D printing. The print time was about 12 hours, which was less than the time estimated by CURA. After the printing was completed, we removed the toy from the heated print bed. Next, we removed any supporting material on the
Figure 2: The CAD Drawing Is Pre-processed Before It Is 3D Printed Using Cura, A Slicing Software

Table 1: 3D Printing Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fill density for toy</td>
<td>80 %</td>
</tr>
<tr>
<td>Fill density for support material</td>
<td>10%</td>
</tr>
<tr>
<td>Scaling (CAD:3D-printing)</td>
<td>3:4</td>
</tr>
<tr>
<td>Nozzle temperature</td>
<td>220 °C</td>
</tr>
</tbody>
</table>

body that surrounded the moving leg, using a combination of a screwdriver and manual shaking. The toy is able to walk downhill on a 7.87 degree when given a slight perturbation.

Additionally, we have explored painting the toy to correspond to the UTSA brand identity colors. The final colored toy is shown in Fig. 3. The main reason for painting the toy was the unavailability of a PLA spool corresponding to UTSA brand identity colors. Acrylic paint is normally used to paint PLA plastic. We used acrylic to paint the white portion of the toy. Due to the non-availability of UTSA brand blue and orange colors in acrylic, we used Valspar signature semi-gloss interior high-hiding paint and primer. First, we cleaned the toy of all the support material using needle nose pliers and needle files. Then, we sanded by hand all of the sharp edges with a fine grit sand paper. We removed the sanding dust from the crevices of the plastic using a toothbrush dipped in dish soap. The toy was washed and then left to dry. Once fully dry, we painted the toy by applying 5 layers of Valspar’s paint due to the strong initial absorption of the PLA. We allowed the paint to dry out before painting successive layers to avoid clumping of the paint. The sanding and hand painting took about 10 hours. However, by using a sheet sander and an airbrush, we were able to reduce the time to less than 2 hours. Finally, the toy was sealed with 2 coats of Valspar premium finish paint and primer Micromist finishing spray.
3 Modelling Details

A caricature of the model with the dimensions and frames of reference is shown in Fig. 4. A single step of the walker is shown in Fig. 5 and is referenced in the equation below

(a) Rear-leg Stop
(b,c) One DOF, Front Leg Stance
(d) Front-to-rear Support
(e) Two DOF, Rear Leg Stance
(f) Front-leg Stop

(1)

Transition conditions are above the arrow and phases of motion are between the arrows. For example, One DOF, Front Leg Stance indicates that the walker moves as a single unit (i.e., the swing leg is locked to the stance leg due the angle stop). Then the transition condition, Front-to-rear Support occurs wherein the support is transferred from the body+front-leg to the rear-leg.
This transition leads to phase *Two DOF, Rear Leg Stance*. In this phase, the rear-leg is the stance leg and rolls freely and the body+front-leg is the swing leg that pivots about the rear-leg – a two degree of freedom system.

![Figure 5: A single step of the walker: See Eq.(1) for more details.](image)

Equations of motion

Next, we describe the derivation of the equations for each phase and transition. We reference the transitions/phases in the order shown in Eq. (1) and Fig. 5. The equations were derived using the *Symbolic Toolbox* in MATLAB. All MATLAB files are provided on github [1].

3.1 One DOF, Front Leg Stance (see Fig. 5 (b,c))

In this phase, the body+front-leg is the stance leg (see Fig. 6 (a)) and the rear-leg rests against the stance leg. Thus, the dynamics of the system are due to $q_1$, whereas $q_2 = u_2 = 0$. The equations
are

\[ A\ddot{q}_1 = b, \]
\[ \dot{q}_2 = 0. \]

The first equation used angular momentum balance about the contact point \( \vec{H}/P = \vec{M}/P \) and second equation is because the swing leg is stationary relative to the stance leg. The \( A \) and \( b \) matrices are

\[ A = 2c_1 m_1 r \cos(q_1) - c_2^2 m_2 - m_1 r^2 - m_2 r^2 - m_1 w_1^2 - m_2 w_2^2 - c_1^2 m_1 + 2 m_1 r w_1 \sin(q_1) + 2 c_2 m_2 r \cos(q_1 - q_2) + 2 m_2 r w_2 \sin(q_1 - q_2) - I_1 - I_2 \]

\[ b = g m_1 r \sin(\gamma) - g m_2 r \cos(\gamma - q_1 + q_2) - c_2 g m_2 \sin(\gamma - q_1 + q_2) + g m_2 r \sin(\gamma) - c_1 g m_1 \sin(\gamma - q_1) - g m_1 w_1 \cos(\gamma - q_1) + c_1 m_1 r u_1^2 \sin(q_1) - m_1 r u_1^2 w_1 \cos(q_1) + c_2 m_2 r u_1^2 \sin(q_1 - q_2) - m_2 r u_1^2 w_2 \cos(q_1 - q_2) \]

The total energy at the beginning of this phase is found using the formula and using appropriate values for the state.

\[ E = \frac{I_1 u_1^2}{2} + \frac{I_2 u_1^2}{2} + \frac{m_2 \left( u_1^2 \left( c_1 \sin(q_1) - w_1 \cos(q_1) \right)^2 + (r u_1 - u_1 \left( c_2 \cos(q_1 - q_2) + w_2 \sin(q_1 - q_2) \right))^2 \right)}{2} \]

\[ + \frac{m_1 \left( u_1^2 \left( c_1 \sin(q_1) - w_1 \cos(q_1) \right)^2 + (r u_1 - u_1 \left( c_1 \cos(q_1) + w_1 \sin(q_1) \right))^2 \right)}{2} \]

\[ - g m_2 \left( c_2 \cos(q_1 - q_2) - r + w_2 \sin(q_1 - q_2) \right) \]

\[ - \sin(\gamma) \left( q_1 r - c_2 \sin(q_1 - q_2) + w_2 \cos(q_1 - q_2) \right) \]

\[ + g m_1 \left( \sin(\gamma) \left( q_1 r - c_1 \sin(q_1) + w_1 \cos(q_1) \right) - \cos(\gamma) \left( c_1 \cos(q_1) + r + w_1 \sin(q_1) \right) \right) \]

### 3.2 Front-to-rear Support Transition (see Fig. 5 (d))

The support transfer from body+front-leg to rear-leg occurs when the line joining the pin joint to the rear edge of the front leg is vertical, that is, along direction of gravity. Mathematically, the transition condition is \( q_1 = \gamma \). The transition is smooth (i.e., there is no instantaneous change in angular velocity) because both legs have the same length and curvature. Because \( q_1 \) and \( q_2 \) denotes the position of the stance leg and swing leg respectively, we need to swap the roles of the legs. The conditions are

\[ q_1^+ = q_1^- - q_2^-, \quad u_1^+ = u_1^- - u_2^-, \quad \text{(2)} \]
\[ q_2^+ = -q_2^- = 0, \quad u_2^+ = -u_2^- = 0. \quad \text{(3)} \]

### 3.3 Two DOF, Rear Leg Stance (see Fig. 5 (e))

In this phase, the rear-leg is the stance leg and the body+front-leg is the swing leg. We obtain two equations by use angular momentum balance about the contact point, \( \vec{H}/P = \vec{M}/P \), and about the
pin joint, \( \vec{H}/H = \vec{M}/H_P \) (see Fig. 6(b)). The equations are of the form

\[ \mathbf{Aq} = \mathbf{b}, \]

where \( \mathbf{q} = \{q_1, q_2\} \), \( \mathbf{A} \) is a 2x2 matrix, and \( \mathbf{b} \) is a 2x1 vector. If \( A_{i,j} \) is the element on the \( i \)th row and \( j \)th of \( \mathbf{A} \), and if \( b_i \) is the \( i \)th row of \( \mathbf{b} \) then

\[
A_{1,1} = -I_1 - I_2 - 2c_2 m_2 r \cos(q_1) - c_2^2 m_2 - m_1 r^2 - m_2 r^2 - m_1 w_1^2 - m_2 w_2^2 \\
- c_1^2 m_1 + 2 m_2 r w_2 \sin(q_1) + 2 c_1 m_1 r \cos(q_1 - q_2) + 2 m_1 r w_1 \sin(q_1 - q_2),
\]

\[
A_{1,2} = A_{2,1},
\]

\[
A_{2,2} = m_1 (c_1^2 + w_1^2) + I_1,
\]

\[
b_1 = c_1 u_1 - c_1 g m_1 \sin(\gamma - q_1 + q_2) - g m_1 w_1 \cos(\gamma - q_1 + q_2) + g m_1 r \sin(\gamma) + g m_2 r \sin(\gamma) \\
- c_2 g m_2 \sin(\gamma - q_1) - g m_2 w_2 \cos(\gamma - q_1) + c_2 m_2 r u_1^2 \sin(q_1) - m_2 r u_1^2 w_2 \cos(q_1) \\
+ c_1 m_1 r u_1^2 \sin(q_1 - q_2) + c_1 m_1 r u_2^2 \sin(q_1 - q_2) - m_1 r u_1^2 w_1 \cos(q_1 - q_2) \\
- m_1 r u_2^2 w_1 \cos(q_1 - q_2) - 2 c_1 m_1 r u_1 u_2 \sin(q_1 - q_2) + 2 m_1 r u_1 u_2 w_1 \cos(q_1 - q_2),
\]

\[
b_2 = g m_1 (\cos(\gamma) (c_1 \sin(q_1 - q_2) - w_1 \cos(q_1 - q_2)) - \sin(\gamma) (c_1 \cos(q_1 - q_2) + w_1 \sin(q_1 - q_2))) .
\]

We have added a viscous damping term, \( C_1 \times u_1 \) to \( b_1 \), to dampen unstable rocking motion of the rear leg. The total energy at the beginning of this phase is found using the formula and using appropriate values for the state.

\[
E = \frac{m_1 \left( ((c_1 \cos(q_1 - q_2) + w_1 \sin(q_1 - q_2)) (u_1 - u_2) - r u_1)^2 \right)}{2} \\
+ \frac{m_1 \left( (c_1 \sin(q_1 - q_2) - w_1 \cos(q_1 - q_2))^2 (u_1 - u_2)^2 \right)}{2} + \frac{I_2 u_1^2}{2} + \frac{I_1 (u_1 - u_2)^2}{2} \\
+ \frac{m_2 \left( u_1^2 (c_2 \sin(q_1) - w_2 \cos(q_1))^2 + (r u_1 - u_1 (c_2 \cos(q_1) + w_2 \sin(q_1)))^2 \right)}{2} \\
- g m_1 (\cos(\gamma) (c_1 \cos(q_1 - q_2) - r + w_1 \sin(q_1 - q_2)) \\
- \sin(\gamma) (q_1 r - c_1 \sin(q_1 - q_2) + w_1 \cos(q_1 - q_2))) \\
+ g m_2 (\sin(\gamma) (q_1 r - c_2 \sin(q_1) + w_2 \cos(q_1)) - \cos(\gamma) (c_2 \cos(q_1) - r + w_2 \sin(q_1))) .
\]

### 3.4 Front-leg Stop Transition (see Fig. 5 (f))

A mechanical stop restricts the angle between the legs to \( \alpha \). The front-leg stop transition occurs when the swing leg, body+front-leg, reaches the angle limit, i.e., \( q_2 = -\alpha \). The stance leg angle remains unchanged between the transition, but the stance leg velocity undergoes a discrete change.
The stance leg velocity after transition is found using conservation of angular momentum about the pin joint H (see Fig. 6 (b)), \( \vec{H}_H=\vec{H}_H' \). The equations are

\[
\begin{align*}
q_1^+ &= q_1^- , \\
q_2^+ &= -\alpha , \\
Au_1^+ &= b , \\
u_2^+ &= 0 .
\end{align*}
\]

where \( A \) and \( b \) are given as follows

\[
A = I_1 + I_2 + m_2 \left( (c_2 \sin(q_1^+) - w_2 \cos(q_1^+))^2 + (c_2 \cos(q_1^+) - r + w_2 \sin(q_1^+))^2 \right) \\
+ m_1 \left( (c_1 \cos(q_1^+ - q_2^+) - r + w_1 \sin(q_1^+ - q_2^+))^2 + (c_1 \sin(q_1^+ - q_2^+) - w_1 \cos(q_1^+ - q_2^+))^2 \right)
\]

\[
b = m_1 \left( ((c_1 \cos(u_1^- - q_2^-) + w_1 \sin(q_1^- - q_2^-)) (u_1^- - u_2^-) - r u_1^-) (c_1 \cos(q_1^- - q_2^-) - r \\
+ w_1 \sin(q_1^- - q_2^-)) + (c_1 \sin(q_1^- - q_2^-) - w_1 \cos(q_1^- - q_2^-))^2 (u_1^- - u_2^-) \right) \\
+ m_2 \left( u_1^- (c_2 \cos(q_1^-) - r + w_2 \sin(q_1^-))^2 + u_1^- (c_2 \sin(q_1^-) - w_2 \cos(q_1^-))^2 \right) \\
+ I_2 u_1^- + I_1 (u_1^- - u_2^-)
\]

3.5 One DOF, Rear Leg Stance (see Fig. 5 (g))

In this phase, the rear-leg is the stance leg (see Fig. 6 (b)) and the body+front-leg is held at fixed angle to the stance leg due to the mechanical stop. Thus, the dynamics of the system are due to \( q_1 \), whereas \( q_2 = -\alpha \) and \( u_2 = 0 \). The equations are

\[
A\ddot{q}_1 = b , \\
\ddot{q}_2 = 0 .
\]

The first equation used angular momentum balance about the contact point \( \vec{H}_{FP} = \vec{M}_{FP} \) and second equation is because the swing leg is stationary relative to the stance leg. The \( A \) and \( b \) matrices are

\[
A = -I_1 - I_2 + 2 c_1 m_1 r \cos(q_1) - c_2^2 m_2 - m_1 r^2 - m_2 r^2 - m_1 w_1^2 - m_2 w_2^2 \\
- c_1^2 m_1 + 2 m_1 r w_1 \sin(q_1) + 2 c_2 m_2 r \cos(q_1 - q_2) + 2 m_2 r w_2 \sin(q_1 - q_2)
\]

\[
b = g m_1 r \sin(\gamma) - g m_2 w_2 \cos(\gamma - q_1 + q_2) - c_2 g m_2 \sin(\gamma - q_1 + q_2) \\
+ g m_2 r \sin(\gamma) - c_1 g m_1 \sin(\gamma - q_1) - g m_1 w_1 \cos(\gamma - q_1) + c_1 m_1 r u_1^2 \sin(q_1) \\
- m_1 r u_1^2 w_1 \cos(q_1) + c_2 m_2 r u_1^2 \sin(q_1 - q_2) - m_2 r u_1^2 w_2 \cos(q_1 - q_2)
\]
The total energy at the beginning of this phase is found using the formula and using appropriate values for the state. Note that the formula is the same as $E_{b,c}$:

$$E = \frac{I_1 u_1^2}{2} + \frac{I_2 u_1^2}{2} + m_1 \left( u_1^2 (c_1 \sin(q_1 - q_2) - w_1 \cos(q_1 - q_2))^2 + (r u_1 - u_1 (c_1 \cos(q_1 - q_2) + w_1 \sin(q_1 - q_2)))^2 \right)$$

$$+ \frac{m_2 \left( u_1^2 (c_2 \sin(q_1) - w_2 \cos(q_1))^2 + (r u_1 - u_1 (c_2 \cos(q_1) + w_2 \sin(q_1)))^2 \right)}{2}$$

$$- g m_1 (\cos(\gamma) (c_1 \cos(q_1 - q_2) - r + w_1 \sin(q_1 - q_2))$$

$$- \sin(\gamma) (q_1 r - c_1 \sin(q_1 - q_2) + w_1 \cos(q_1 - q_2)))$$

$$+ g m_2 (\sin(\gamma) (q_1 r - c_2 \sin(q_1) + w_2 \cos(q_1)) - \cos(\gamma) (c_2 \cos(q_1) - r + w_2 \sin(q_1))))$$

### 3.6 Rear-to-front Support Transition (see Fig. 5 (h))

We assume that the support transfer from rear-leg to body+front-leg occurs when the axis of the rear leg are vertical. Mathematically, the condition is given as follows, $q_1 - q_2 = \gamma$. The transition is smooth (i.e., there is no instantaneous change in angular velocity) because both legs have the same length and curvature. Because $q_1$ and $q_2$ denotes the position of the stance leg and swing leg respectively, we need to swap the roles of the legs. The conditions are

$$q_1^+ = q_1^- - q_2^-, \quad u_1^+ = u_1^- - u_2^-,$$

$$q_2^+ = -q_2^- = \alpha, \quad u_2^+ = -u_2^- = 0. \quad (6) \quad (7)$$

### 3.7 Two DOF, Front Leg Stance (see Fig. 5 (i))

In this phase, the body+front-leg is the stance leg (see Fig. 6 (a)) and the rear-leg is the swing leg. We obtain two equations by use angular momentum balance about the contact point, $\vec{H}/P = \vec{M}/P,$ and about the pin joint, $\vec{H}/H = \vec{M}/HP$ (see Fig. 6 (a)). The equations are of the form

$$A \ddot{q} = b,$$

where $q = \{q_1, q_2\}$, $A$ is a 2x2 matrix, and $b$ is a 2x1 vector. If $A_{i,j}$ is the element on the $i$th row and $j$th of $A$, and if $b_i$ is the $i$th row of $b$ then

$$A_{1,1} = -I_1 - I_2 + 2 c_1 m_1 r \cos(q_1) - c_2^2 m_2 - m_1 r^2 - m_2 r^2 - m_1 w_1^2 - m_2 w_2^2$$

$$- c_1^2 m_1 + 2 m_1 r w_1 \sin(q_1) + 2 c_2 m_2 r \cos(q_1 - q_2) + 2 m_2 r w_2 \sin(q_1 - q_2)$$

$$A_{1,2} = -A_{2,1}$$

$$= I_2 + m_2 (c_2^2 - r \cos(q_1 - q_2) c_2 + w_2^2 - r \sin(q_1 - q_2) w_2)$$

$$A_{2,2} = I_2 + m_2 (c_2^2 + w_2^2)$$

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\[ b_1 = -m_2 \left( (c_2 \sin(q_1 - q_2) - w_2 \cos(q_1 - q_2)) (u_1 - u_2)^2 (c_2 \cos(q_1 - q_2) - r + w_2 \sin(q_1 - q_2)) \\
- (c_2 \cos(q_1 - q_2) + w_2 \sin(q_1 - q_2)) (c_2 \sin(q_1 - q_2) - w_2 \cos(q_1 - q_2)) (u_1 - u_2)^2 \right) \\
- m_1 \left( u_1^2 (c_1 \sin(q_1) - w_1 \cos(q_1)) (c_1 \cos(q_1) - r + w_1 \sin(q_1)) \\
- u_1^2 (c_1 \cos(q_1) + w_1 \sin(q_1)) (c_1 \sin(q_1) - w_1 \cos(q_1)) \right) \\
- g m_1 (\sin(\gamma)) (c_1 \cos(q_1) - r + w_1 \sin(q_1)) - \cos(\gamma) (c_1 \sin(q_1) - w_1 \cos(q_1)) \\
- g m_2 (\cos(\gamma)) (c_2 \cos(q_1 - q_2) - r + w_2 \sin(q_1 - q_2)) - \cos(\gamma) (c_2 \sin(q_1 - q_2) - w_2 \cos(q_1 - q_2)) \]

\[ b_2 = g m_2 (\cos(\gamma)) (c_2 \sin(q_1 - q_2) - w_2 \cos(q_1 - q_2)) - \sin(\gamma) (c_2 \cos(q_1 - q_2) + w_2 \sin(q_1 - q_2)) \]

The total energy at the beginning of this phase is found using the formula and using appropriate values for the state.

\[ E = \frac{m_2}{2} \left( (c_2 \cos(q_1 - q_2) + w_2 \sin(q_1 - q_2)) (u_1 - u_2) - r u_1 \right)^2 \\
+ \frac{m_2}{2} \left( (c_2 \sin(q_1 - q_2) - w_2 \cos(q_1 - q_2))^2 (u_1 - u_2)^2 \right) + \frac{I_1 u_1^2}{2} + \frac{I_2 (u_1 - u_2)^2}{2} \\
+ \frac{m_1}{2} \left( u_1^2 (c_1 \sin(q_1) - w_1 \cos(q_1))^2 + (r u_1 - u_1 (c_1 \cos(q_1) + w_1 \sin(q_1)))^2 \right) \\
- g m_2 (\cos(\gamma)) (c_2 \cos(q_1 - q_2) - r + w_2 \sin(q_1 - q_2)) \\
- \sin(\gamma) (q_1 r - c_2 \sin(q_1 - q_2) + w_2 \cos(q_1 - q_2)) \\
+ g m_1 (\sin(\gamma)) (q_1 r - c_1 \sin(q_1) + w_1 \cos(q_1)) - \cos(\gamma) (c_1 \cos(q_1) - r + w_1 \sin(q_1)) \]

### 3.8 Rear-leg Stop Transition (see Fig. 5(j))

The rear-leg stop transition occurs when the swing leg, rear-leg collides with the stance leg, the condition is \( q_2 = 0 \). The stance leg angle remains unchanged between the transition, but the stance leg velocity undergoes a discrete change. The stance leg velocity after transition is found using conservation of angular momentum about the pin joint \( H \) (see Fig. 6(a)), \( \vec{H}_H = \vec{H}/H \). The equations are

\[ q_1^+ = q_1^- \]
\[ A u_1^+ = b, \]
\[ q_2^+ = 0, \]
\[ u_2^+ = 0. \]

where \( A \) and \( b \) are given as follows

\[ A = I_1 + I_2 + m_1 \left( (c_1 \sin(q_1^+) - w_1 \cos(q_1^+))^2 + (c_1 \cos(q_1^+) - r + w_1 \sin(q_1^+))^2 \right) \\
+ m_2 \left( (c_2 \cos(q_1^+ - q_2^+) - r + w_2 \sin(q_1^+ - q_2^+))^2 + (c_2 \sin(q_1^+ - q_2^+) - w_2 \cos(q_1^+ - q_2^+))^2 \right) \]

\[ b = m_2 \left( ((c_2 \cos(q_1^- - q_2) + w_2 \sin(q_1^- - q_2^-)) (u_1^- - u_2^-) - r u_1^-) (c_2 \cos(q_1^- - q_2) - r \\
+ w_2 \sin(q_1^- - q_2^-)) + (c_2 \sin(q_1^- - q_2^-) - w_2 \cos(q_1^- - q_2^-))^2 (u_1^- - u_2^-) \right) \\
+ m_1 \left( u_1^- (c_1 \cos(q_1^-) - r + w_1 \sin(q_1^-))^2 + u_1^- (c_1 \sin(q_1^-) - w_1 \cos(q_1^-))^2 \right) \\
+ I_1 u_1^- + I_2 (u_1^- - u_2^-) \]
References
