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Supplementary Information

for the paper

A 3D Printed, Non-Assembly, Passive Dynamic Walking Toy: Design and Analysis

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1 Notation

1.1 Robot parameters

Symbol	Value	Parameter description
m_1	141 gm	Mass of body.
m_2	16 gm	Mass of rear leg.
I_1	1300 $gm - cm^2$	Inertia of body.
I_2	30 $gm - cm^2$	Inertia of rear leg.
r	5.5 cm	Leg length.
c_1	1.76 cm	Distance of body COM along leg axis (positive is downward).
w_1	0.75×10^{-3} cm	Distance of body COM normal to leg axis (positive is backward).
c_2	3.85 cm	Distance of rear leg COM along leg axis (positive is downward).
w_2	0.71×10^{-3} cm	Distance of rear leg COM normal to leg axis (positive is backward).
g	9.81 m/s^2	Gravitational constant.
γ	0.1372 rad	Ramp slope.
α	0.2182 rad	Maximum angle between the legs.
C_1	0.005 Ns/m	Viscous friction between rear leg and ground.

1.2 Simulation variables

Symbol	Variable description
t	time.
q_1	absolute angle between the stance leg and normal to the ramp.
q_2	relative angle between the swing leg and the stance leg.

u_1 angular velocity of the stance leg ($= \dot{q}_1$).
u_2 angular velocity of the swing leg ($= \dot{q}_2$).
q_1^+, q_1^- absolute angle of the stance leg wrt. vertical after and before transition.
u_1^+, u_1^- angular velocity of the stance leg wrt. vertical after and before transition.
q_2^+, q_2^- relative angle of the swing leg wrt. stance leg after and before transition.
u_2^+, u_2^- angular velocity of the swing leg wrt. stance leg after and before transition.
$\vec{H}^+_{/P}$ $\vec{H}^-_{/P}$ Angular momentum about point P after and before transition.
$\dot{\vec{H}}_{/P}$ Rate of change of angular momentum about point P and so on.
$\vec{M}_{/P}$ External angular moment about point P and so on.

2 Toy Design and 3D printing

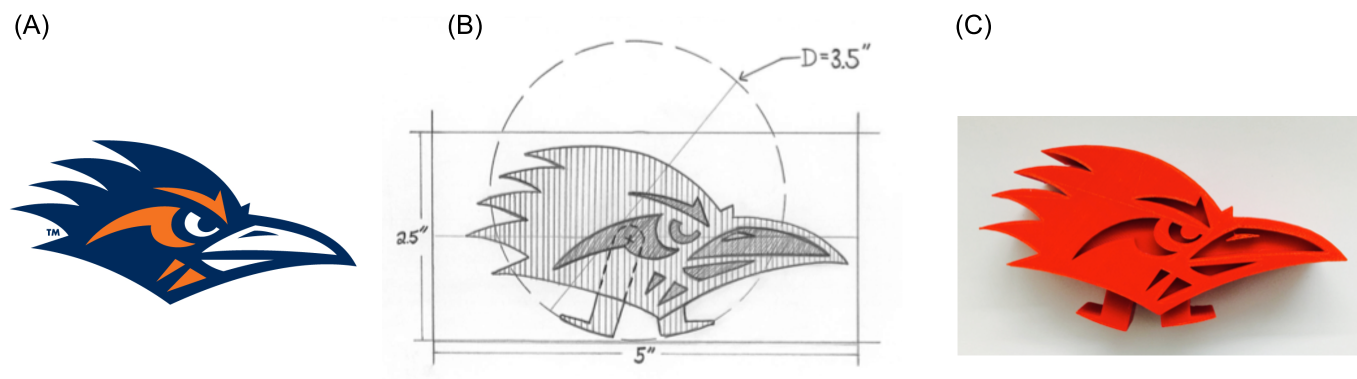

Figure 1: (A) UTSA Mascot "Rowdy" The Roadrunner Logo [2], (B) a sketch of the toy, and (C) a CAD rendering of the toy

The Fig. 1 (A) shows the UTSA mascot from the University's Communications and Marketing website. The goal of the design stage was to add legs to the mascot, followed by tuning the mass distribution, leg geometry, and the tolerances on the hinge joint to create a functional walking prototype. The Fig. 1 (B) shows the initial sketch of the toy with legs and Fig. 1 (C) shows a 3D CAD rendering of the sketch that was drawn using SOLIDWORKS.

Next, the toy was pre-processed using CURA, an open source slicing software. CURA divides the CAD drawing into a number of slices as shown in Fig. 2. We also needed to specify other 3D printing options at this stage. The specifications included the: scaling, temperature of the nozzle, material fill for the toy and/or the support material. Note that we can only choose the in-fill for one of the two, toy or the support material and the other in-fill is automatically determined by CURA. These specifications are given in Tab. 1. CURA estimated the print time to be 15 hours.

Next the toy was printed using an Ultimaker 2, a desktop hobby-grade 3D printer. We used a 2.85mm diameter Polyactic Acid (PLA) filament for 3D printing. The print time was about 12 hours, which was less than the time estimated by CURA. After the printing was completed, we removed the toy from the heated print bed. Next, we removed any supporting material on the

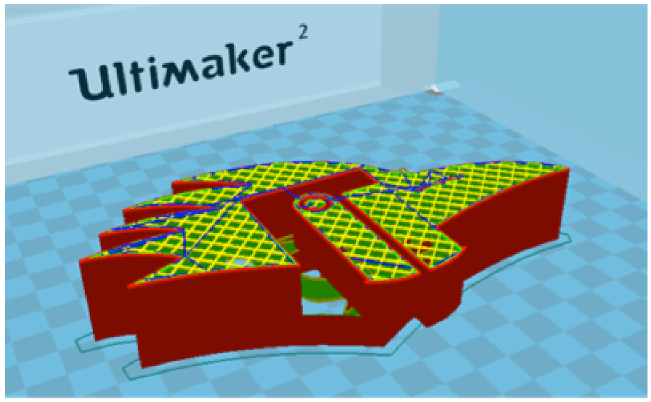

Figure 2: The CAD Drawing Is Pre-processed Before It Is 3D Printed Using Cura, A Slicing Software

Table 1: 3D Printing Parameters

Parameter	Value
Fill density for toy	80 %
Fill density for support material	10%.
Scaling (CAD:3D-printing)	3:4
Nozzle temperature	220 °C

body that surrounded the moving leg, using a combination of a screwdriver and manual shaking. The toy is able to walk downhill on a 7.87 degree when given a slight perturbation.

Additionally, we have explored painting the toy to correspond to the UTSA brand identity colors. The final colored toy is shown in Fig. 3. The main reason for painting the toy was the unavailability of a PLA spool corresponding to UTSA brand identity colors. Acrylic paint is normally used to paint PLA plastic. We used acrylic to paint the white portion of the toy. Due to the non-availability of UTSA brand blue and orange colors in acrylic, we used Valspar signature semi-gloss interior high-hiding paint and primer. First, we cleaned the toy of all the support material using needle nose pliers and needle files. Then, we sanded by hand all of the sharp edges with a fine grit sand paper. We removed the sanding dust from the crevices of the plastic using a toothbrush dipped in dish soap. The toy was washed and then left to dry. Once fully dry, we painted the toy by applying 5 layers of Valspar's paint due to the strong initial absorption of the PLA. We allowed the paint to dry out before painting successive layers to avoid clumping of the paint. The sanding and hand painting took about 10 hours. However, by using a sheet sander and an airbrush, we were able to reduce the time to less than 2 hours. Finally, the toy was sealed with 2 coats of Valspar premium finish paint and primer Micromist finishing spray.

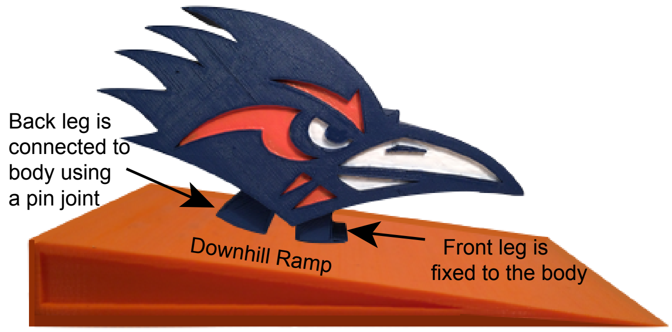

Figure 3: The Final Product

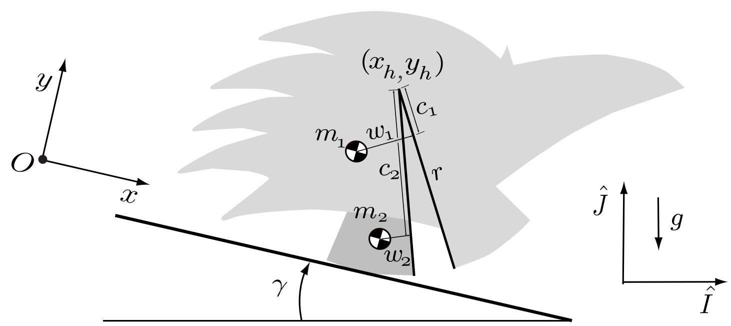

Figure 4: **The model:** The robot consists of two pieces: (1) The body and front leg (light gray) is a single unit (we designate this piece as the body + front-leg), and (2) the rear leg (dark gray). The rear leg connects to body + front-leg by means of a pin joint.

3 Modelling Details

A caricature of the model with the dimensions and frames of reference is shown in Fig. 4. A single step of the walker is shown in Fig. 5 and is referenced in the equation below

$$\begin{aligned}
&\overbrace{\longrightarrow}^{\text{(a) Rear-leg Stop}} \underbrace{\text{(b,c) One DOF, Front Leg Stance} \overbrace{\longrightarrow}^{\text{(d) Front-to-rear Support}} \text{(e) Two DOF, Rear Leg Stance} \overbrace{\longrightarrow}^{\text{(f) Front-leg Stop}}}_{\text{single step (line 1)}} \\
&\underbrace{\text{(g) One DOF, Rear Leg Stance} \overbrace{\longrightarrow}^{\text{(h) Rear-to-front Support}} \text{(i) Two DOF, Front Leg Stance} \overbrace{\longrightarrow}^{\text{(j) Rear-leg Stop}}}_{\text{single step (line 2)}}
\end{aligned} \tag{1}$$

Transition conditions are above the arrow and phases of motion are between the arrows. For example, *One DOF, Front Leg Stance* indicates that the walker moves as a single unit (i.e., the swing leg is locked to the stance leg due the angle stop). Then the transition condition, *Front-to-rear Support* occurs wherein the support is transferred from the body+front-leg to the rear-leg.

This transition leads to phase *Two DOF, Rear Leg Stance*. In this phase, the rear-leg is the stance leg and rolls freely and the body+front-leg is the swing leg that pivots about the rear-leg – a two degree of freedom system.

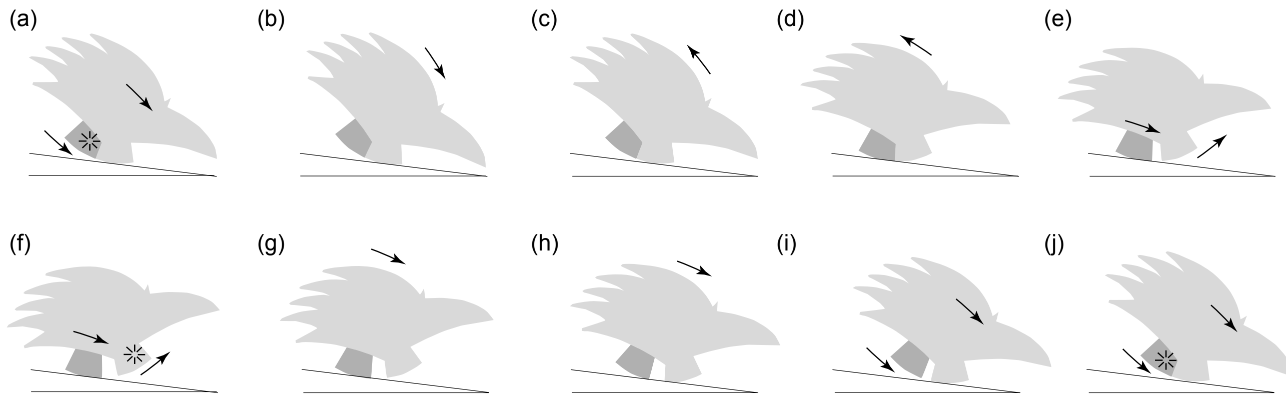

Figure 5: **A single step of the walker:** See Eq.(1) for more details.

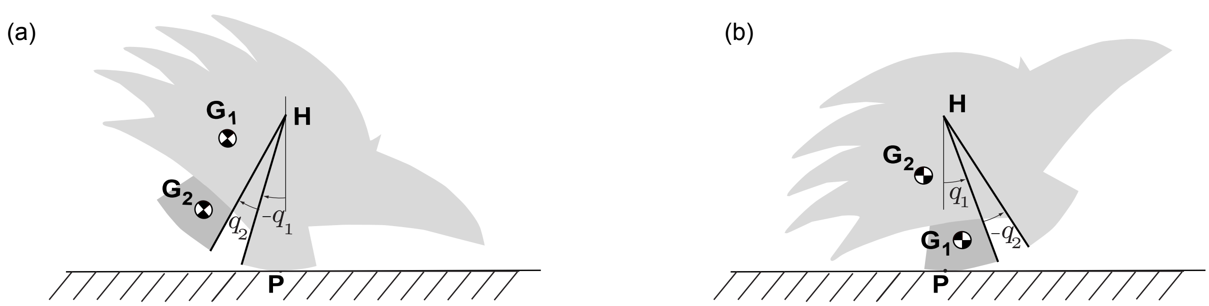

Figure 6: **Angles and naming convention for deriving the equations for the two modes:** (a) Front leg is the stance leg. (b) Rear leg is the stance leg. P is the contact point with the ground.

Equations of motion

Next, we describe the derivation of the equations for each phase and transition. We reference the transitions/phases in the order shown in Eq. (1) and Fig. 5. The equations were derived using the *Symbolic Toolbox* in MATLAB. All MATLAB files are provided on github [1].

3.1 One DOF, Front Leg Stance (see Fig. 5 (b,c))

In this phase, the body+front-leg is the stance leg (see Fig. 6 (a)) and the rear-leg rests against the stance leg. Thus, the dynamics of the system are due to q_1, whereas $q_2 = u_2 = 0$. The equations are

$$A\ddot{q}_1 = b,$$
$$\ddot{q}_2 = 0.$$

The first equation used angular momentum balance about the contact point $\dot{\vec{H}}_{/P} = \vec{M}_{/P}$ and second equation is because the swing leg is stationary relative to the stance leg. The A and b matrices are

$$\begin{aligned} A = {} & 2\,\mathrm{c_1}\,\mathrm{m_1}\,r\,\cos(\mathrm{q_1}) - \mathrm{c_2}^2\,\mathrm{m_2} - \mathrm{m_1}\,r^2 - \mathrm{m_2}\,r^2 - \mathrm{m_1}\,\mathrm{w_1}^2 - \mathrm{m_2}\,\mathrm{w_2}^2 - \mathrm{c_1}^2\,\mathrm{m_1} \\ & + 2\,\mathrm{m_1}\,r\,\mathrm{w_1}\,\sin(\mathrm{q_1}) + 2\,\mathrm{c_2}\,\mathrm{m_2}\,r\,\cos(\mathrm{q_1}-\mathrm{q_2}) + 2\,\mathrm{m_2}\,r\,\mathrm{w_2}\,\sin(\mathrm{q_1}-\mathrm{q_2}) - I_1 - I_2 \end{aligned}$$

$$\begin{aligned} b = {} & g\,\mathrm{m_1}\,r\,\sin(\gamma) - g\,\mathrm{m_2}\,\mathrm{w_2}\,\cos(\gamma-\mathrm{q_1}+\mathrm{q_2}) - \mathrm{c_2}\,g\,\mathrm{m_2}\,\sin(\gamma-\mathrm{q_1}+\mathrm{q_2}) \\ & + g\,\mathrm{m_2}\,r\,\sin(\gamma) - \mathrm{c_1}\,g\,\mathrm{m_1}\,\sin(\gamma-\mathrm{q_1}) - g\,\mathrm{m_1}\,\mathrm{w_1}\,\cos(\gamma-\mathrm{q_1}) + \mathrm{c_1}\,\mathrm{m_1}\,r\,\mathrm{u_1}^2\,\sin(\mathrm{q_1}) \\ & - \mathrm{m_1}\,r\,\mathrm{u_1}^2\,\mathrm{w_1}\,\cos(\mathrm{q_1}) + \mathrm{c_2}\,\mathrm{m_2}\,r\,\mathrm{u_1}^2\,\sin(\mathrm{q_1}-\mathrm{q_2}) - \mathrm{m_2}\,r\,\mathrm{u_1}^2\,\mathrm{w_2}\,\cos(\mathrm{q_1}-\mathrm{q_2}) \end{aligned}$$

The total energy at the beginning of this phase is found using the formula and using appropriate values for the state.

$$\begin{aligned} E = {} & \frac{\mathrm{I_1}\,\mathrm{u_1}^2}{2} + \frac{\mathrm{I_2}\,\mathrm{u_1}^2}{2} \\ & + \frac{\mathrm{m_2}\,\left(\mathrm{u_1}^2\,(\mathrm{c_2}\,\sin(\mathrm{q_1}-\mathrm{q_2}) - \mathrm{w_2}\,\cos(\mathrm{q_1}-\mathrm{q_2}))^2 + (r\,\mathrm{u_1} - \mathrm{u_1}\,(\mathrm{c_2}\,\cos(\mathrm{q_1}-\mathrm{q_2}) + \mathrm{w_2}\,\sin(\mathrm{q_1}-\mathrm{q_2})))^2\right)}{2} \\ & + \frac{\mathrm{m_1}\,\left(\mathrm{u_1}^2\,(\mathrm{c_1}\,\sin(\mathrm{q_1}) - \mathrm{w_1}\,\cos(\mathrm{q_1}))^2 + (r\,\mathrm{u_1} - \mathrm{u_1}\,(\mathrm{c_1}\,\cos(\mathrm{q_1}) + \mathrm{w_1}\,\sin(\mathrm{q_1})))^2\right)}{2} \\ & - g\,\mathrm{m_2}\,(\cos(\gamma)\,(\mathrm{c_2}\,\cos(\mathrm{q_1}-\mathrm{q_2}) - r + \mathrm{w_2}\,\sin(\mathrm{q_1}-\mathrm{q_2})) \\ & \qquad - \sin(\gamma)\,(\mathrm{q_1}\,r - \mathrm{c_2}\,\sin(\mathrm{q_1}-\mathrm{q_2}) + \mathrm{w_2}\,\cos(\mathrm{q_1}-\mathrm{q_2}))) \\ & + g\,\mathrm{m_1}\,(\sin(\gamma)\,(\mathrm{q_1}\,r - \mathrm{c_1}\,\sin(\mathrm{q_1}) + \mathrm{w_1}\,\cos(\mathrm{q_1})) - \cos(\gamma)\,(\mathrm{c_1}\,\cos(\mathrm{q_1}) - r + \mathrm{w_1}\,\sin(\mathrm{q_1}))) \end{aligned}$$

3.2 Front-to-rear Support Transition (see Fig. 5 (d))

The support transfer from body+front-leg to rear-leg occurs when the line joining the pin joint to the rear edge of the front leg is vertical, that is, along direction of gravity. Mathematically, the transition condition is $q_1 = \gamma$. The transition is smooth (i.e., there is no instantaneous change in angular velocity) because both legs have the same length and curvature. Because q_1 and q_2 denotes the position of the stance leg and swing leg respectively, we need to swap the roles of the legs. The conditions are

$$q_1^+ = q_1^- - q_2^-, \qquad u_1^+ = u_1^- - u_2^-, \tag{2}$$
$$q_2^+ = -q_2^- = 0, \qquad u_2^+ = -u_2^- = 0. \tag{3}$$

3.3 Two DOF, Rear Leg Stance (see Fig. 5 (e))

In this phase, the rear-leg is the stance leg and the body+front-leg is the swing leg. We obtain two equations by use angular momentum balance about the contact point, $\dot{\vec{H}}_{/P} = \vec{M}_{/P}$, and about the pin joint, $\dot{\vec{H}}_{/H} = \vec{M}_{/HP}$ (see Fig. 6 (b)). The equations are of the form

$$\mathbf{A}\ddot{\mathbf{q}} = \mathbf{b},$$

where $\mathbf{q} = \{q_1,\ q_2\}$, $\mathbf{A}$ is a 2x2 matrix, and $\mathbf{b}$ is a 2x1 vector. If $A_{i,j}$ is the element on the ith row and jth of $\mathbf{A}$, and if b_i is the ith row of $\mathbf{b}$ then

$$\begin{aligned} A_{1,1} = {} & -I_1 - I_2 - 2\,\mathrm{c_2}\,\mathrm{m_2}\,r\,\cos(\mathrm{q_1}) - \mathrm{c_2}^2\,\mathrm{m_2} - \mathrm{m_1}\,r^2 - \mathrm{m_2}\,r^2 - \mathrm{m_1}\,\mathrm{w_1}^2 - \mathrm{m_2}\,\mathrm{w_2}^2 \\ & - \mathrm{c_1}^2\,\mathrm{m_1} + 2\,\mathrm{m_2}\,r\,\mathrm{w_2}\,\sin(\mathrm{q_1}) + 2\,\mathrm{c_1}\,\mathrm{m_1}\,r\,\cos(\mathrm{q_1}-\mathrm{q_2}) + 2\,\mathrm{m_1}\,r\,\mathrm{w_1}\,\sin(\mathrm{q_1}-\mathrm{q_2})\,, \end{aligned}$$

$$\begin{aligned} A_{1,2} &= A_{2,1}, \\ &= I_1 + \mathrm{m_1}\,\left(\mathrm{c_1}^2 - 1\,r\,\cos(\mathrm{q_1}-\mathrm{q_2})\,\mathrm{c_1} + \mathrm{w_1}^2 - 1\,r\,\sin(\mathrm{q_1}-\mathrm{q_2})\,\mathrm{w_1}\right), \end{aligned}$$

$$A_{2,2} = \mathrm{m_1}\,\left(\mathrm{c_1}^2 + \mathrm{w_1}^2\right) + \mathrm{I_1},$$

$$\begin{aligned} b_1 = {} & \mathrm{c_1}\,\mathrm{u_1} - \mathrm{c_1}\,g\,\mathrm{m_1}\,\sin(\gamma - \mathrm{q_1} + \mathrm{q_2}) - g\,\mathrm{m_1}\,\mathrm{w_1}\,\cos(\gamma - \mathrm{q_1} + \mathrm{q_2}) + g\,\mathrm{m_1}\,r\,\sin(\gamma) + g\,\mathrm{m_2}\,r\,\sin(\gamma) \\ & - \mathrm{c_2}\,g\,\mathrm{m_2}\,\sin(\gamma - \mathrm{q_1}) - g\,\mathrm{m_2}\,\mathrm{w_2}\,\cos(\gamma - \mathrm{q_1}) + \mathrm{c_2}\,\mathrm{m_2}\,r\,\mathrm{u_1}^2\,\sin(\mathrm{q_1}) - \mathrm{m_2}\,r\,\mathrm{u_1}^2\,\mathrm{w_2}\,\cos(\mathrm{q_1}) \\ & + \mathrm{c_1}\,\mathrm{m_1}\,r\,\mathrm{u_1}^2\,\sin(\mathrm{q_1}-\mathrm{q_2}) + \mathrm{c_1}\,\mathrm{m_1}\,r\,\mathrm{u_2}^2\,\sin(\mathrm{q_1}-\mathrm{q_2}) - \mathrm{m_1}\,r\,\mathrm{u_1}^2\,\mathrm{w_1}\,\cos(\mathrm{q_1}-\mathrm{q_2}) \\ & - \mathrm{m_1}\,r\,\mathrm{u_2}^2\,\mathrm{w_1}\,\cos(\mathrm{q_1}-\mathrm{q_2}) - 2\,\mathrm{c_1}\,\mathrm{m_1}\,r\,\mathrm{u_1}\,\mathrm{u_2}\,\sin(\mathrm{q_1}-\mathrm{q_2}) + 2\,\mathrm{m_1}\,r\,\mathrm{u_1}\,\mathrm{u_2}\,\mathrm{w_1}\,\cos(\mathrm{q_1}-\mathrm{q_2})\,, \end{aligned}$$

$$b_2 = g\,\mathrm{m_1}\,\left(\cos(\gamma)\,\left(\mathrm{c_1}\,\sin(\mathrm{q_1}-\mathrm{q_2}) - \mathrm{w_1}\,\cos(\mathrm{q_1}-\mathrm{q_2})\right) - \sin(\gamma)\,\left(\mathrm{c_1}\,\cos(\mathrm{q_1}-\mathrm{q_2}) + \mathrm{w_1}\,\sin(\mathrm{q_1}-\mathrm{q_2})\right)\right).$$

We have added a viscous damping term, $C_1 \times u_1$ to b_1, to dampen unstable rocking motion of the rear leg. The total energy at the beginning of this phase is found using the formula and using appropriate values for the state.

$$\begin{aligned} E = {} & \frac{\mathrm{m_1}\,\left(\left(\left(\mathrm{c_1}\,\cos(\mathrm{q_1}-\mathrm{q_2}) + \mathrm{w_1}\,\sin(\mathrm{q_1}-\mathrm{q_2})\right)\,(\mathrm{u_1}-\mathrm{u_2}) - r\,\mathrm{u_1}\right)^2\right)}{2} \\ & + \frac{\mathrm{m_1}\,\left(\left(\mathrm{c_1}\,\sin(\mathrm{q_1}-\mathrm{q_2}) - \mathrm{w_1}\,\cos(\mathrm{q_1}-\mathrm{q_2})\right)^2\,(\mathrm{u_1}-\mathrm{u_2})^2\right)}{2} + \frac{\mathrm{I_2}\,\mathrm{u_1}^2}{2} + \frac{\mathrm{I_1}\,(\mathrm{u_1}-\mathrm{u_2})^2}{2} \\ & + \frac{\mathrm{m_2}\,\left(\mathrm{u_1}^2\,\left(\mathrm{c_2}\,\sin(\mathrm{q_1}) - \mathrm{w_2}\,\cos(\mathrm{q_1})\right)^2 + \left(r\,\mathrm{u_1} - \mathrm{u_1}\,\left(\mathrm{c_2}\,\cos(\mathrm{q_1}) + \mathrm{w_2}\,\sin(\mathrm{q_1})\right)\right)^2\right)}{2} \\ & - g\,\mathrm{m_1}\,\left(\cos(\gamma)\,\left(\mathrm{c_1}\,\cos(\mathrm{q_1}-\mathrm{q_2}) - r + \mathrm{w_1}\,\sin(\mathrm{q_1}-\mathrm{q_2})\right)\right. \\ & \qquad \left. - \sin(\gamma)\,\left(\mathrm{q_1}\,r - \mathrm{c_1}\,\sin(\mathrm{q_1}-\mathrm{q_2}) + \mathrm{w_1}\,\cos(\mathrm{q_1}-\mathrm{q_2})\right)\right) \\ & + g\,\mathrm{m_2}\,\left(\sin(\gamma)\,\left(\mathrm{q_1}\,r - \mathrm{c_2}\,\sin(\mathrm{q_1}) + \mathrm{w_2}\,\cos(\mathrm{q_1})\right) - \cos(\gamma)\,\left(\mathrm{c_2}\,\cos(\mathrm{q_1}) - r + \mathrm{w_2}\,\sin(\mathrm{q_1})\right)\right) \end{aligned}$$

3.4 Front-leg Stop Transition (see Fig. 5 (f))

A mechanical stop restricts the angle between the legs to α. The front-leg stop transition occurs when the swing leg, body+front-leg, reaches the angle limit, i.e., $q_2 = -\alpha$. The stance leg angle remains unchanged between the transition, but the stance leg velocity undergoes a discrete change.

The stance leg velocity after transition is found using conservation of angular momentum about the pin joint H (see Fig. 6 (b)), $\vec{H}^{+}_{/H} = \vec{H}^{-}_{/H}$. The equations are

$$q_1^+ = q_1^-, \qquad Au_1^+ = b, \tag{4}$$
$$q_2^+ = -\alpha, \qquad u_2^+ = 0. \tag{5}$$

where A and b are given as follows

$$\begin{aligned} A = I_1 + I_2 + \mathrm{m}_2 \left(\left(\mathrm{c}_2 \sin(\mathrm{q}_1^+) - \mathrm{w}_2 \cos(\mathrm{q}_1^+)\right)^2 + \left(\mathrm{c}_2 \cos(\mathrm{q}_1^+) - r + \mathrm{w}_2 \sin(\mathrm{q}_1^+)\right)^2 \right) \\ + \mathrm{m}_1 \left(\left(\mathrm{c}_1 \cos(\mathrm{q}_1^+ - \mathrm{q}_2^+) - r + \mathrm{w}_1 \sin(\mathrm{q}_1^+ - \mathrm{q}_2^+)\right)^2 + \left(\mathrm{c}_1 \sin(\mathrm{q}_1^+ - \mathrm{q}_2^+) - \mathrm{w}_1 \cos(\mathrm{q}_1^+ - \mathrm{q}_2^+)\right)^2 \right) \end{aligned}$$

$$\begin{aligned} b = \mathrm{m}_1 \Big(\left(\left(\mathrm{c}_1 \cos(\mathrm{u}_1^- - \mathrm{q}_2^-) + \mathrm{w}_1 \sin(\mathrm{q}_1^- - \mathrm{q}_2^-)\right) (\mathrm{u}_1^- - \mathrm{u}_2^-) - r\,\mathrm{u}_1^- \right) \left(\mathrm{c}_1 \cos(\mathrm{q}_1^- - \mathrm{q}_2^-) - r \right. \\ \left. + \mathrm{w}_1 \sin(\mathrm{q}_1^- - \mathrm{q}_2^-)\right) + \left(\mathrm{c}_1 \sin(\mathrm{q}_1^- - \mathrm{q}_2^-) - \mathrm{w}_1 \cos(\mathrm{q}_1^- - \mathrm{q}_2^-)\right)^2 (\mathrm{u}_1^- - \mathrm{u}_2^-) \Big) \\ + \mathrm{m}_2 \left(\mathrm{u}_1^- \left(\mathrm{c}_2 \cos(\mathrm{q}_1^-) - r + \mathrm{w}_2 \sin(\mathrm{q}_1^-)\right)^2 + \mathrm{u}_1^- \left(\mathrm{c}_2 \sin(\mathrm{q}_1^-) - \mathrm{w}_2 \cos(\mathrm{q}_1^-)\right)^2 \right) \\ + I_2 u_1^- + I_1(u_1^- - u_2^-) \end{aligned}$$

3.5 One DOF, Rear Leg Stance (see Fig. 5 (g))

In this phase, the rear-leg is the stance leg (see Fig. 6 (b)) and the body+front-leg is held at fixed angle to the stance leg due to the mechanical stop. Thus, the dynamics of the system are due to q_1, whereas $q_2 = -\alpha$ and $u_2 = 0$. The equations are

$$A\ddot{q}_1 = b,$$
$$\ddot{q}_2 = 0.$$

The first equation used angular momentum balance about the contact point $\dot{\vec{H}}_{/P} = \vec{M}_{/P}$ and second equation is because the swing leg is stationary relative to the stance leg. The A and b matrices are

$$\begin{aligned} A = -I_1 - I_2 + 2\,\mathrm{c}_1\,\mathrm{m}_1\,r\,\cos(\mathrm{q}_1) - {\mathrm{c}_2}^2\,\mathrm{m}_2 - \mathrm{m}_1\,r^2 - \mathrm{m}_2\,r^2 - \mathrm{m}_1\,{\mathrm{w}_1}^2 - \mathrm{m}_2\,{\mathrm{w}_2}^2 \\ - {\mathrm{c}_1}^2\,\mathrm{m}_1 + 2\,\mathrm{m}_1\,r\,\mathrm{w}_1\,\sin(\mathrm{q}_1) + 2\,\mathrm{c}_2\,\mathrm{m}_2\,r\,\cos(\mathrm{q}_1 - \mathrm{q}_2) + 2\,\mathrm{m}_2\,r\,\mathrm{w}_2\,\sin(\mathrm{q}_1 - \mathrm{q}_2) \end{aligned}$$

$$\begin{aligned} b = g\,\mathrm{m}_1\,r\,\sin(\gamma) - g\,\mathrm{m}_2\,\mathrm{w}_2\,\cos(\gamma - \mathrm{q}_1 + \mathrm{q}_2) - \mathrm{c}_2\,g\,\mathrm{m}_2\,\sin(\gamma - \mathrm{q}_1 + \mathrm{q}_2) \\ + g\,\mathrm{m}_2\,r\,\sin(\gamma) - \mathrm{c}_1\,g\,\mathrm{m}_1\,\sin(\gamma - \mathrm{q}_1) - g\,\mathrm{m}_1\,\mathrm{w}_1\,\cos(\gamma - \mathrm{q}_1) + \mathrm{c}_1\,\mathrm{m}_1\,r\,{\mathrm{u}_1}^2\,\sin(\mathrm{q}_1) \\ - \mathrm{m}_1\,r\,{\mathrm{u}_1}^2\,\mathrm{w}_1\,\cos(\mathrm{q}_1) + \mathrm{c}_2\,\mathrm{m}_2\,r\,{\mathrm{u}_1}^2\,\sin(\mathrm{q}_1 - \mathrm{q}_2) - \mathrm{m}_2\,r\,{\mathrm{u}_1}^2\,\mathrm{w}_2\,\cos(\mathrm{q}_1 - \mathrm{q}_2) \end{aligned}$$

The total energy at the beginning of this phase is found using the formula and using appropriate values for the state. Note that the formula is the same as $E_{b,c}$.

$$
\begin{aligned}
E =& \frac{\mathrm{I}_1\, \mathrm{u}_1^{\,2}}{2} + \frac{\mathrm{I}_2\, \mathrm{u}_1^{\,2}}{2} \\
&+ \frac{\mathrm{m}_1 \left(\mathrm{u}_1^{\,2} \left(\mathrm{c}_1 \sin(\mathrm{q}_1-\mathrm{q}_2) - \mathrm{w}_1 \cos(\mathrm{q}_1-\mathrm{q}_2)\right)^2 + \left(r\, \mathrm{u}_1 - \mathrm{u}_1 \left(\mathrm{c}_1 \cos(\mathrm{q}_1-\mathrm{q}_2) + \mathrm{w}_1 \sin(\mathrm{q}_1-\mathrm{q}_2)\right)\right)^2\right)}{2} \\
&+ \frac{\mathrm{m}_2 \left(\mathrm{u}_1^{\,2} \left(\mathrm{c}_2 \sin(\mathrm{q}_1) - \mathrm{w}_2 \cos(\mathrm{q}_1)\right)^2 + \left(r\, \mathrm{u}_1 - \mathrm{u}_1 \left(\mathrm{c}_2 \cos(\mathrm{q}_1) + \mathrm{w}_2 \sin(\mathrm{q}_1)\right)\right)^2\right)}{2} \\
&- g\, \mathrm{m}_1 \left(\cos(\gamma) \left(\mathrm{c}_1 \cos(\mathrm{q}_1-\mathrm{q}_2) - r + \mathrm{w}_1 \sin(\mathrm{q}_1-\mathrm{q}_2)\right)\right. \\
&\qquad \left. - \sin(\gamma) \left(\mathrm{q}_1\, r - \mathrm{c}_1 \sin(\mathrm{q}_1-\mathrm{q}_2) + \mathrm{w}_1 \cos(\mathrm{q}_1-\mathrm{q}_2)\right)\right) \\
&+ g\, \mathrm{m}_2 \left(\sin(\gamma) \left(\mathrm{q}_1\, r - \mathrm{c}_2 \sin(\mathrm{q}_1) + \mathrm{w}_2 \cos(\mathrm{q}_1)\right) - \cos(\gamma) \left(\mathrm{c}_2 \cos(\mathrm{q}_1) - r + \mathrm{w}_2 \sin(\mathrm{q}_1)\right)\right)
\end{aligned}
$$

3.6 Rear-to-front Support Transition (see Fig. 5 (h))

We assume that the support transfer from rear-leg to body+front-leg occurs when the axis of the rear leg are vertical. Mathematically, the condition is given as follows, $q_1 - q_2 = \gamma$. The transition is smooth (i.e., there is no instantaneous change in angular velocity) because both legs have the same length and curvature. Because q_1 and q_2 denotes the position of the stance leg and swing leg respectively, we need to swap the roles of the legs. The conditions are

$$q_1^+ = q_1^- - q_2^-, \qquad u_1^+ = u_1^- - u_2^-, \tag{6}$$

$$q_2^+ = -q_2^- = \alpha, \qquad u_2^+ = -u_2^- = 0. \tag{7}$$

3.7 Two DOF, Front Leg Stance (see Fig. 5 (i))

In this phase, the body+front-leg is the stance leg (see Fig. 6 (a)) and the rear-leg is the swing leg. We obtain two equations by use angular momentum balance about the contact point, $\dot{\vec{H}}_{/P} = \vec{M}_{/P}$, and about the pin joint, $\dot{\vec{H}}_{/H} = \vec{M}_{/HP}$ (see Fig. 6 (a)). The equations are of the form

$$\mathbf{A}\ddot{\mathbf{q}} = \mathbf{b},$$

where $\mathbf{q} = \{q_1,\ q_2\}$, $\mathbf{A}$ is a 2x2 matrix, and $\mathbf{b}$ is a 2x1 vector. If $A_{i,j}$ is the element on the ith row and jth of $\mathbf{A}$, and if b_i is the ith row of $\mathbf{b}$ then

$$
\begin{aligned}
A_{1,1} = & -I_1 - I_2 + 2\, \mathrm{c}_1\, \mathrm{m}_1\, r \cos(\mathrm{q}_1) - \mathrm{c}_2^{\,2}\, \mathrm{m}_2 - \mathrm{m}_1\, r^2 - \mathrm{m}_2\, r^2 - \mathrm{m}_1\, \mathrm{w}_1^{\,2} - \mathrm{m}_2\, \mathrm{w}_2^{\,2} \\
& - \mathrm{c}_1^{\,2}\, \mathrm{m}_1 + 2\, \mathrm{m}_1\, r\, \mathrm{w}_1 \sin(\mathrm{q}_1) + 2\, \mathrm{c}_2\, \mathrm{m}_2\, r \cos(\mathrm{q}_1-\mathrm{q}_2) + 2\, \mathrm{m}_2\, r\, \mathrm{w}_2 \sin(\mathrm{q}_1-\mathrm{q}_2)
\end{aligned}
$$

$$
\begin{aligned}
A_{1,2} &= -A_{2,1} \\
&= I_2 + \mathrm{m}_2 \left(\mathrm{c}_2^{\,2} - r \cos(\mathrm{q}_1-\mathrm{q}_2)\, \mathrm{c}_2 + \mathrm{w}_2^{\,2} - r \sin(\mathrm{q}_1-\mathrm{q}_2)\, \mathrm{w}_2\right)
\end{aligned}
$$

$$A_{2,2} = I_2 + \mathrm{m}_2 \left(\mathrm{c}_2^{\,2} + \mathrm{w}_2^{\,2}\right)$$

$$b_1 = -\mathrm{m}_2 \Big((\mathrm{c}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2) - \mathrm{w}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2))\ (\mathrm{u}_1 - \mathrm{u}_2)^2\ (\mathrm{c}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2) - r + \mathrm{w}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2))$$
$$- (\mathrm{c}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2) + \mathrm{w}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2))\ (\mathrm{c}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2) - \mathrm{w}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2))\ (\mathrm{u}_1 - \mathrm{u}_2)^2 \Big)$$
$$- \mathrm{m}_1 \left({\mathrm{u}_1}^2\ (\mathrm{c}_1 \sin(\mathrm{q}_1) - \mathrm{w}_1 \cos(\mathrm{q}_1))\ (\mathrm{c}_1 \cos(\mathrm{q}_1) - r + \mathrm{w}_1 \sin(\mathrm{q}_1)) \right.$$
$$\left. - {\mathrm{u}_1}^2\ (\mathrm{c}_1 \cos(\mathrm{q}_1) + \mathrm{w}_1 \sin(\mathrm{q}_1))\ (\mathrm{c}_1 \sin(\mathrm{q}_1) - \mathrm{w}_1 \cos(\mathrm{q}_1)) \right)$$
$$- g\, \mathrm{m}_1\ (\sin(\gamma)\ (\mathrm{c}_1 \cos(\mathrm{q}_1) - r + \mathrm{w}_1 \sin(\mathrm{q}_1)) - \cos(\gamma)\ (\mathrm{c}_1 \sin(\mathrm{q}_1) - \mathrm{w}_1 \cos(\mathrm{q}_1)))$$
$$- g\, \mathrm{m}_2\ (\sin(\gamma)\ (\mathrm{c}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2) - r + \mathrm{w}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2)) - \cos(\gamma)\ (\mathrm{c}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2) - \mathrm{w}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2)))$$

$$b_2 = g\, \mathrm{m}_2\ (\cos(\gamma)\ (\mathrm{c}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2) - \mathrm{w}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2)) - \sin(\gamma)\ (\mathrm{c}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2) + \mathrm{w}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2)))$$

The total energy at the beginning of this phase is found using the formula and using appropriate values for the state.

$$E = \frac{\mathrm{m}_2 \left(((\mathrm{c}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2) + \mathrm{w}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2))\ (\mathrm{u}_1 - \mathrm{u}_2) - r\, \mathrm{u}_1)^2 \right)}{2}$$
$$+ \frac{\mathrm{m}_2 \left((\mathrm{c}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2) - \mathrm{w}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2))^2\, (\mathrm{u}_1 - \mathrm{u}_2)^2 \right)}{2} + \frac{\mathrm{I}_1\, {\mathrm{u}_1}^2}{2} + \frac{\mathrm{I}_2\, (\mathrm{u}_1 - \mathrm{u}_2)^2}{2}$$
$$+ \frac{\mathrm{m}_1 \left({\mathrm{u}_1}^2\, (\mathrm{c}_1 \sin(\mathrm{q}_1) - \mathrm{w}_1 \cos(\mathrm{q}_1))^2 + (r\, \mathrm{u}_1 - \mathrm{u}_1\ (\mathrm{c}_1 \cos(\mathrm{q}_1) + \mathrm{w}_1 \sin(\mathrm{q}_1)))^2 \right)}{2}$$
$$- g\, \mathrm{m}_2\ (\cos(\gamma)\ (\mathrm{c}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2) - r + \mathrm{w}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2))$$
$$- \sin(\gamma)\ (\mathrm{q}_1\, r - \mathrm{c}_2 \sin(\mathrm{q}_1 - \mathrm{q}_2) + \mathrm{w}_2 \cos(\mathrm{q}_1 - \mathrm{q}_2)))$$
$$+ g\, \mathrm{m}_1\ (\sin(\gamma)\ (\mathrm{q}_1\, r - \mathrm{c}_1 \sin(\mathrm{q}_1) + \mathrm{w}_1 \cos(\mathrm{q}_1)) - \cos(\gamma)\ (\mathrm{c}_1 \cos(\mathrm{q}_1) - r + \mathrm{w}_1 \sin(\mathrm{q}_1)))$$

3.8 Rear-leg Stop Transition (see Fig. 5 (j))

The rear-leg stop transition occurs when the swing leg, rear-leg collides with the stance leg, the condition is $q_2 = 0$. The stance leg angle remains unchanged between the transition, but the stance leg velocity undergoes a discrete change. The stance leg velocity after transition is found using conservation of angular momentum about the pin joint H (see Fig. 6 (a)), $\vec{H}^{+}_{/H} = \vec{H}^{-}_{/H}$. The equations are

$$q_1^+ = q_1^-, \qquad A u_1^+ = b, \tag{8}$$
$$q_2^+ = 0, \qquad u_2^+ = 0. \tag{9}$$

where A and b are given as follows

$$A = I_1 + I_2 + \mathrm{m}_1 \left(\left(\mathrm{c}_1 \sin(\mathrm{q}_1^+) - \mathrm{w}_1 \cos(\mathrm{q}_1^+)\right)^2 + \left(\mathrm{c}_1 \cos(\mathrm{q}_1^+) - r + \mathrm{w}_1 \sin(\mathrm{q}_1^+)\right)^2 \right)$$
$$+ \mathrm{m}_2 \left(\left(\mathrm{c}_2 \cos(\mathrm{q}_1^+ - \mathrm{q}_2^+) - r + \mathrm{w}_2 \sin(\mathrm{q}_1^+ - \mathrm{q}_2^+)\right)^2 + \left(\mathrm{c}_2 \sin(\mathrm{q}_1^+ - \mathrm{q}_2^+) - \mathrm{w}_2 \cos(\mathrm{q}_1^+ - \mathrm{q}_2^+)\right)^2 \right)$$

$$b = \mathrm{m}_2 \Big(\left((\mathrm{c}_2 \cos(\mathrm{q}_1^- - \mathrm{q}_2^-) + \mathrm{w}_2 \sin(\mathrm{q}_1^- - \mathrm{q}_2^-))\ (\mathrm{u}_1^- - \mathrm{u}_2^-) - r\, \mathrm{u}_1^- \right)\ \left(\mathrm{c}_2 \cos(\mathrm{q}_1^- - \mathrm{q}_2^-) - r \right.$$
$$\left. + \mathrm{w}_2 \sin(\mathrm{q}_1^- - \mathrm{q}_2^-) \right) + \left(\mathrm{c}_2 \sin(\mathrm{q}_1^- - \mathrm{q}_2^-) - \mathrm{w}_2 \cos(\mathrm{q}_1^- - \mathrm{q}_2^-)\right)^2\ (\mathrm{u}_1^- - \mathrm{u}_2^-) \Big)$$
$$+ \mathrm{m}_1 \left(\mathrm{u}_1^-\ \left(\mathrm{c}_1 \cos(\mathrm{q}_1^-) - r + \mathrm{w}_1 \sin(\mathrm{q}_1^-)\right)^2 + \mathrm{u}_1^-\ \left(\mathrm{c}_1 \sin(\mathrm{q}_1^-) - \mathrm{w}_1 \cos(\mathrm{q}_1^-)\right)^2 \right)$$
$$+ I_1 u_1^- + I_2 (u_1^- - u_2^-)$$

References

[1] Pranav Bhounsule. Matlab simulation of a passive dynamic walker. https://github.com/pab47/RowdyWalker, Accessed March 2018.

[2] UTSA. Roadrunner graphic. http://www.utsa.edu/ucm/resources/identity/visual-style/roadrunner.html, Accessed February 2016 2016.