Hyperbolic PDE

First order convection equation

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \qquad -(1)$$

Second order wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\frac{\partial^2 \phi}{\partial x^2} \qquad -(2)$$

For (2) $B = 0$

$A = 1 \quad C = -c^2$

$B^2 - 4AC = 0 - 4(1)(-c^2) > 0$ Hyperbolic

It can be shown that ① reduces to ②

$$\frac{\partial \phi}{\partial t} = -c\frac{\partial \phi}{\partial x} \quad \text{—③}$$

Differentiating both sides with respect to time

$$\frac{\partial^2 \phi}{\partial t^2} = -c\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x}\right) = -c\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t}\right)$$

switching the derivative

$$= -c\frac{\partial}{\partial x}\left(-c\frac{\partial \phi}{\partial x}\right) \quad \text{from ③}$$

$$\frac{\partial^2 \phi}{\partial t^2} = c^2\frac{\partial^2 \phi}{\partial x^2} \quad \text{same as ②}$$

2 initial conditions : $t=0$; $\phi = \alpha_1(x)$

$t=0$; $\frac{\partial \phi}{\partial t} = \alpha_2(x)$

2 boundary conditions : $x=0$; $\phi = \beta_1$ $\quad t>0$

$x=L$; $\phi = \beta_2$ $\quad t>0$

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

ϕ_{ij} $\quad$ $i \equiv$ time , $j \equiv$ space (x)

$$\frac{\overset{?}{\phi_{i+1\,j}} - 2\overset{✓}{\phi_{ij}} + \overset{✓}{\phi_{i-1\,j}}}{\Delta t^2} = c^2 \left[\frac{\overset{✓}{\phi_{i\,j+1}} - 2\overset{✓}{\phi_{ij}} + \overset{✓}{\phi_{i\,j-1}}}{\Delta x^2} \right]$$

Solve for $\phi_{i+1\,j}$

$$\boxed{\phi_{i+1\,j} = -\phi_{\underbrace{i-1,j}_{\text{Corrected}}} + \frac{c^2 \Delta t^2}{\Delta x^2}\left(\phi_{i\,j+1} + \phi_{i\,j-1}\right) + \ldots + 2\left[1 - c^2 \frac{\Delta t^2}{\Delta x^2}\right] \phi_{ij}}$$

Method is stable as long as

$$1 - \frac{c^2 \Delta t^2}{\Delta x^2} \geq 0 \qquad \frac{c \Delta t}{\Delta x} \leq 1$$

Courant Condition

EXAMPLE Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Initial conditions : $u(x,0) = \begin{cases} x & 0 \le x \le 0.5 \\ 1-x & 0.5 < x \le 1 \end{cases}$

$$\frac{du}{dt}(x,0) = 0$$

$\uparrow t=0 \quad i=0$

Boundary conditions : $u(0,t) = u(1,t) = 0$

Use $\Delta x = 0.25$ and $\Delta t = 0.1$.

Solve for u in the range $0 \le x \le 1$ and $0 \le t \le 0.5$

Courant condition

$$\frac{c\,\Delta t}{\Delta x} = \frac{(1)(0.1)}{0.25} = 0.4 < 1$$

Stable!

$$\phi_{i+1\,j} = -\phi_{i-1\,j} + (0.4)^2\left[\phi_{i\,j+1} + \phi_{i\,j-1}\right] + 2(1-0.4^2)\,\phi_{ij}$$

$$U_{i+1\,j} = -U_{i-1\,j} + 0.16\left[U_{i\,j+1} + U_{i\,j-1}\right] + 1.68\,U_{ij}$$

$i \neq 0$ $\quad U_{1j} = -\cancel{U_{-1j}}$

$i=1$ $\quad U_{2j} = -U_{0j} + 0.16\left[U_{1\,j+1} + U_{1\,j-1}\right] + 1.68\,U_{1j}$

$j=1$ $\quad U_{21} = -U_{01} + 0.16\left[U_{12} + U_{10}\right] + 1.68\,U_{11}$

$U_{21} = -0.25 + 0.16\left[0.5 + 0\right] + 1.68\,(0.25)$

$= 0.25$

$j=2$ $\quad U_{22} = -U_{02} + 0.16\left[U_{13} + U_{11}\right] + 1.68\,U_{12}$

$= -0.5 + 0.16\left[0.25 + 0.25\right] + 1.68\,(0.5)$

$= 0.42$

$j=3$ $\quad U_{23} = -U_{03} + 0.16(U_{14} + U_{12}) + 1.68\,U_{13}$

$= -0.25 + 0.16(0 + 0.5) + 1.68\,(0.25)$

$= 0.25$

Solution

i	t \ x (j)	$j=0$: 0	$j=1$: 0.25	$j=2$: 0.5	$j=3$: 0.75	$j=4$: 1
0 (U_{0j})	0	0 (x, V_{00})	0.25 ((x), V_{01})	0.5 ((x), V_{02})	0.25 ($1-x$, V_{03})	0 ($1-x$, U_{04})
1 (U_{ij})	0.1	0 (U_{10})	0.25 (U_{11})	0.5 (U_{12})	0.25 (U_{13})	0 (U_{14})
2 ($i=2$)	0.2	0	0.25	0.42	0.25	0
3	0.3	0	0.2372	0.2856	0.2372	0
4	0.4	0	0.1942	0.1357	0.1942	0
5	0.5	0	0.1108	0.0045	0.1108	0

$U_{1j} = U_{0j}$

$$\frac{du}{dt} = \frac{U_{i+1\,j} - U_{ij}}{\Delta t} = 0 \Rightarrow U_{i+1\,j} = U_{\underset{0}{i}\,j}$$

$$\boxed{U_{1j} = U_{0j}}$$

Solution

	j	0	1	2	3	4
i	t \ x	0	0.25	0.5	0.75	1
0	0	0	0.25	0.5	0.25	0
1	0.1	0	0.25	0.5	0.25	0
2	0.2	0	0.25	0.42	0.25	0
3	0.3	0	0.2372	0.2856	0.2372	0
4	0.4	0	0.1942	0.1357	0.1942	0
5	0.5	0	0.1108	0.0045	0.1108	0