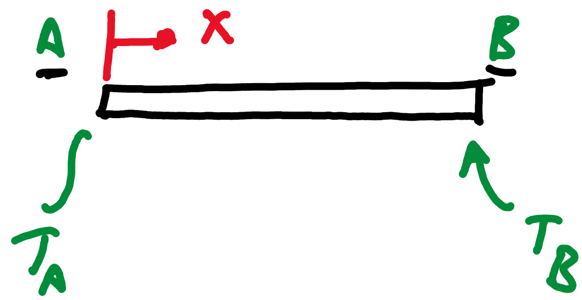
① Parabolic PDE's

1-D Heat conduction

T = temperature

$T(x, t)$

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$



① IC : $T(x, 0) = f(x)$

② BC : $T(a, t) = T_a$

$T(b, t) = T_B$

IC – initial condition
BC – Boundary condition

Goal : To compute $T(x, t)$

$T_{i,j}$

t x

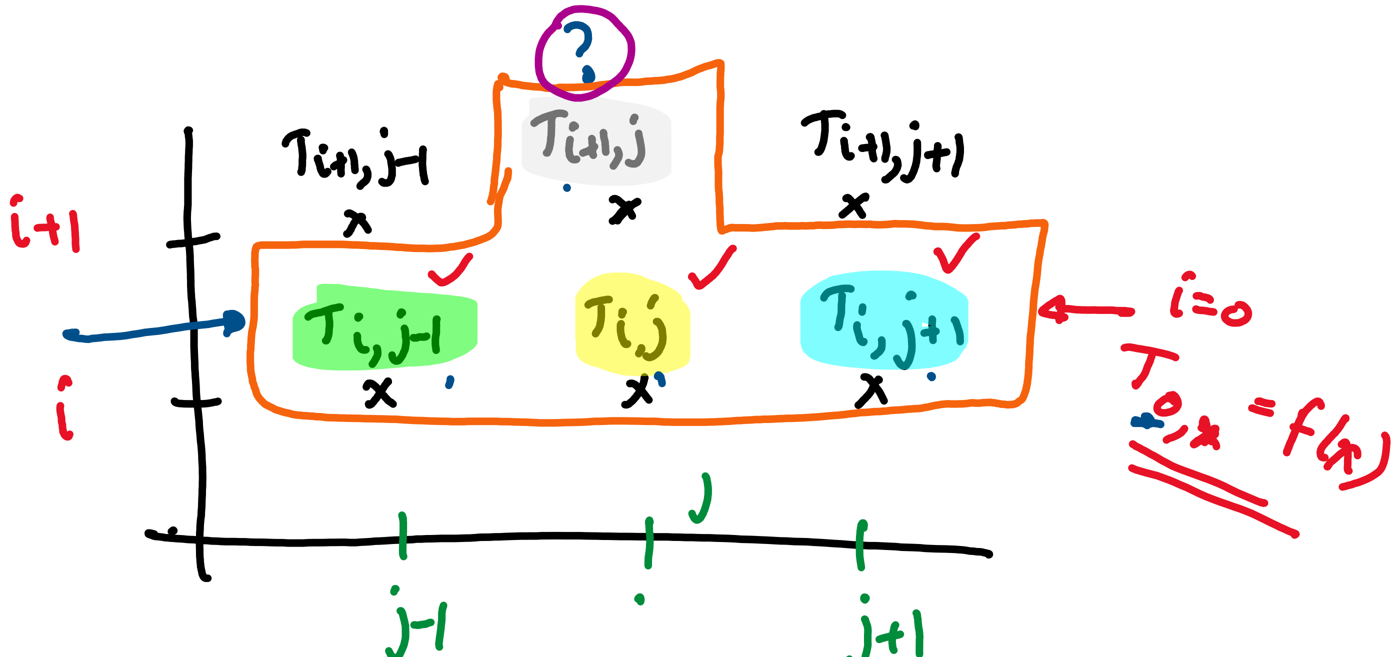
$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

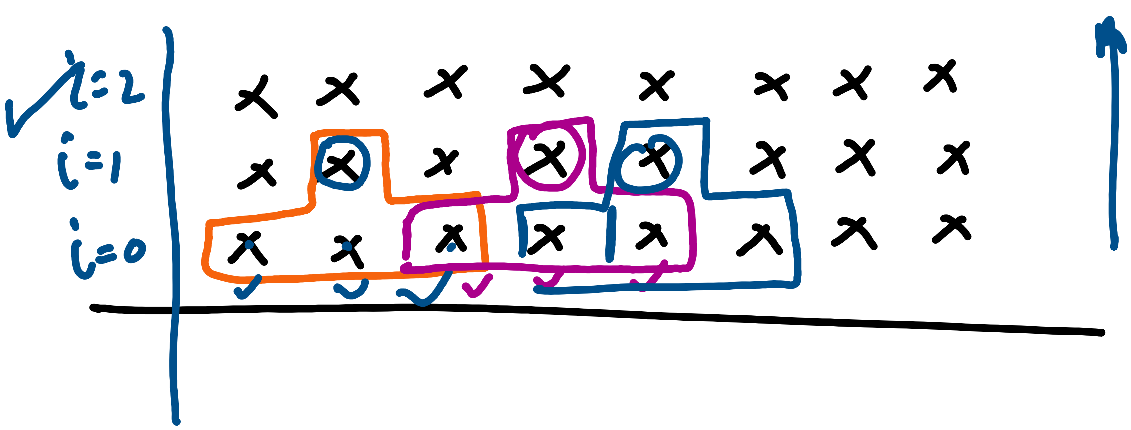
$T_{i,j}$

t x

Finit Difference

$$\alpha \left[\frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta x^2} \right] = \left[\frac{T_{i+1,j} - T_{i,j}}{\Delta t} \right]$$





$$\alpha \left[\overbrace{\frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta x^2}} \right] = \left[\frac{T_{i+1,j} - T_{i,j}}{\Delta t} \right]$$

✓

central diff

Fwd diff

Solve for $T_{i+1,j}$

$$T_{i+1,j} = \left[\frac{\alpha \Delta t}{\Delta x^2}\right] \left(T_{i,j+1} + T_{i,j-1}\right) + \left[1 - \frac{2\alpha \Delta t}{\Delta x^2}\right] T_{i,j}$$

$$\boxed{T_{i+1,j} = F\left(T_{i,j+1} + T_{i,j-1}\right) + [1 - 2F]\, T_{i,j}}$$

(corrected)

where $F = \dfrac{\alpha \Delta t}{\Delta x^2}$

EXPLICIT method

Truncation error $= O(\Delta t) + O(\Delta x^2)$

The method is stable as long as

$$F = \frac{\alpha \Delta t}{\Delta x^2} \leq \frac{1}{2}$$

Forward Time, Central space method (FTCS)

EXAMPLE: $\dfrac{\partial T}{\partial t} = \alpha \dfrac{\partial^2 T}{\partial x^2}$ $(\alpha = 0.2)$

IC: $T(x,0) = 100x(1-x)$

BC: $T(0,t) = T(1,t) = 0$

$\uparrow x=0$ $\quad\quad$ $\uparrow x=1$

Assuming $\Delta x = 0.25$; $\Delta t = 0.1$, compute the temperature profile for

$0 \le t \le 0.5$ and $0 \le x \le 1$

Use Forward Time - Central space method

$$F = \frac{\alpha \Delta t}{\Delta x^2} = \frac{0.2(0.1)}{(0.25)^2} = 0.32 < \frac{1}{2} (=0.5)$$

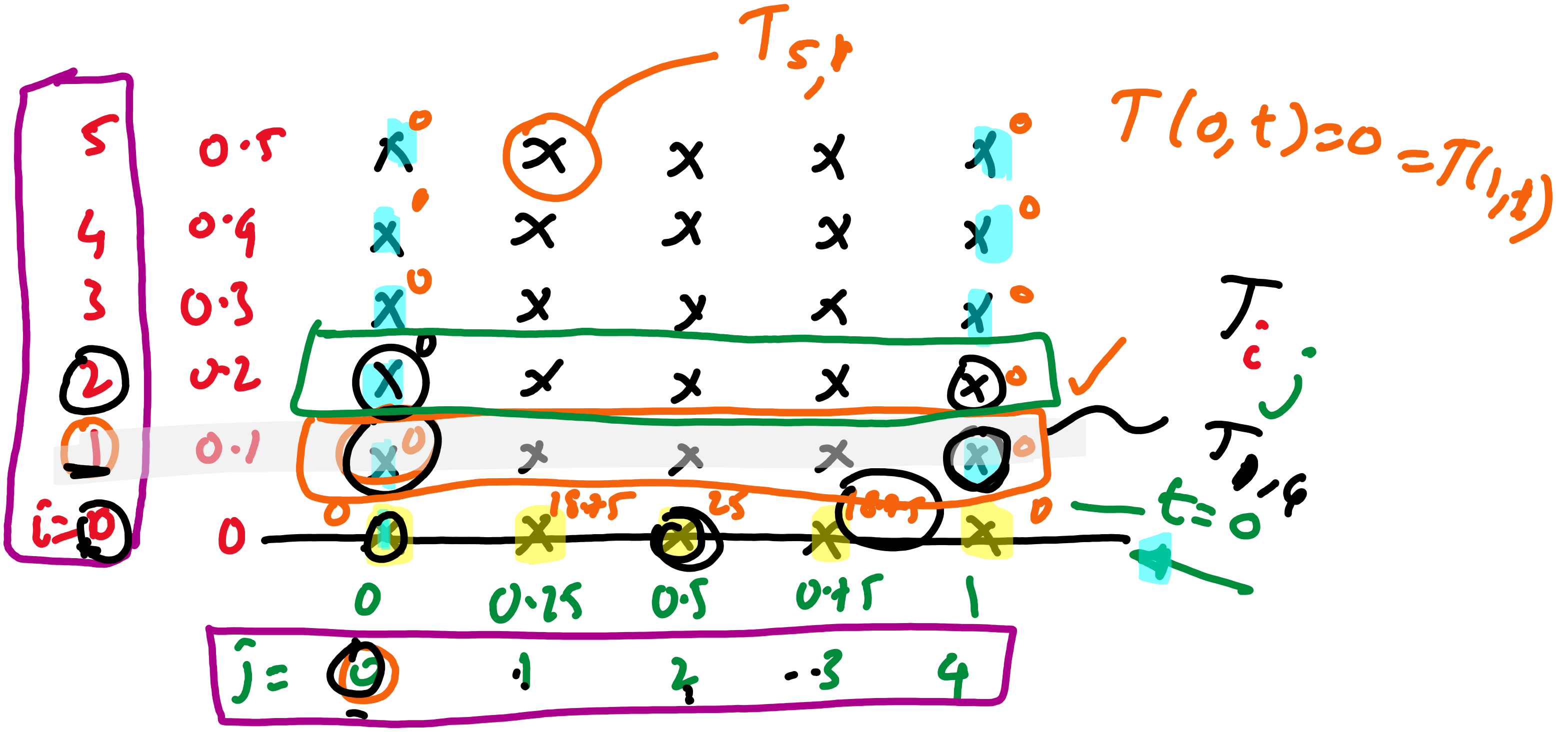
Stability condition satisfied: $F = 0.32 < 0.5$

$t_i = t_0 + i\Delta t = 0, 0.1, 0.2, 0.3, 0.4, 0.5$

$t_0 = 0$; $i = 0, 1, 2, 3, 4, 5$; $\Delta t = 0.1$

$x_j = x_0 + \Delta x\, j = 0, 0.25, 0.5, 0.75, 1$

$x_0 = 0$; $j = 0, 1, 2, 3, 4$; $\Delta x = 0.25$



$$T(x,0) = 100\,x(1-x)$$

$$T(0,0) = 0 \;;$$

$$T(0.25,0) = 100(0.25)(1-0.25) = 18.75$$

$$T(0.5,0) = 100(0.5)(1-0.5) = 25$$

$$T(0.75,0) = 18.75$$

$$T(1,0) = 0$$

FTCS

$$T_{\underset{t}{i+1},\underset{x}{j}} = F\left(T_{i,j+1} + T_{i,j-1}\right) + (1-2F)\,T_{i,j}$$

$i=0$ $\quad T_{1,j} = F\left(T_{0,j+1} + T_{0,j-1}\right) + (1-2F)\,T_{0,j}$

$j=0$ $\quad T_{1,0} = 0 \quad$ (BC)

$j=4$ $\quad T_{1,4} = 0 \quad$ (B.C)

$\underline{j=1}$ $\quad \underline{T_{1,1}} = F\left(T_{0,2} + T_{0,0}\right) + (1-2F)\,T_{0,1}$

$$= 0.32(25+0) + (1-0.32)\,18.75$$

$$= \underline{14.75}$$

$j=2$ $\quad T_{1,2} = F\left(T_{0,3} + T_{0,1}\right) + (1-2F)\,T_{0,2}$

$$\underline{T_{1,2}} = 0.32(18.75+18.75) + (1-2(0.32))\,25$$

$$= \underline{21}$$

$\underline{j=3}$ $\quad T_{1,3} = F\left(T_{0,4} + T_{0,2}\right) + (1-2F)\,T_{0,3}$

$$= 0.32(0+25) + (1-2(0.32)\,18.75$$

$$= 14.75$$

$i=1 \quad T_{2,j} = F(T_{1,j+1} + T_{1,j-1}) + (1-2F)\,T_{1,j}$

$j=0 \quad T_{2,0} = 0$

$j=4 \quad T_{2,4} = 0$

$j=1 \quad T_{2,1} = F(T_{1,2} + T_{1,0}) + (1-2F)\,T_{1,1}$

$= 0.32(21+0) + (1-2(0.32))\,14.75$

$= 12.03$

$j=2 \quad T_{2,2} = F(T_{1,3} + T_{1,1}) + (1-2F)\,T_{1,2}$

$= 0.32(14.75+14.75) + (1-2(0.32))\,21$

$= 17$

$j=3 \quad T_{2,3} = 12.03$

Temperature profile

t \ x	0	0.25	0.5	0.75	1
0	0	18.75	25	18.75	0
0.1	0	14.75	21	14.75	0
0.2	0	12.03	17	12.03	0
0.3	0	9.7708	13.8192	9.7708	0
0.4	0	7.9396	11.2282	7.9396	0
0.5	0	6.4513	9.1235	6.4513	0

Final solution