# Boundary Value Problems (BVP)

A BVP has atleast one condition that is not defined at the initial state $x = x_0$

e.g. $\dfrac{d^2y}{dx^2} = f(x,y)$

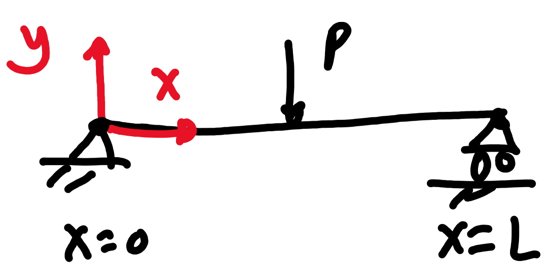
$y(x=x_0) = y_0$

$y(x=x_1) = y_1$ — Boundary Value

$\dfrac{d^2y}{dx^2} = f(x,y)$

$y(x=x_0) = y_0$

$\dfrac{dy}{dx}(x=x_1) = y_1'$




$EI\dfrac{d^2y}{dx} = M(x)$

$y(x=0) = 0$

$y(x=L) = 0$ — Boundary value

## Two methods to solve BVP

(i) Shooting method

(ii) Finite difference method

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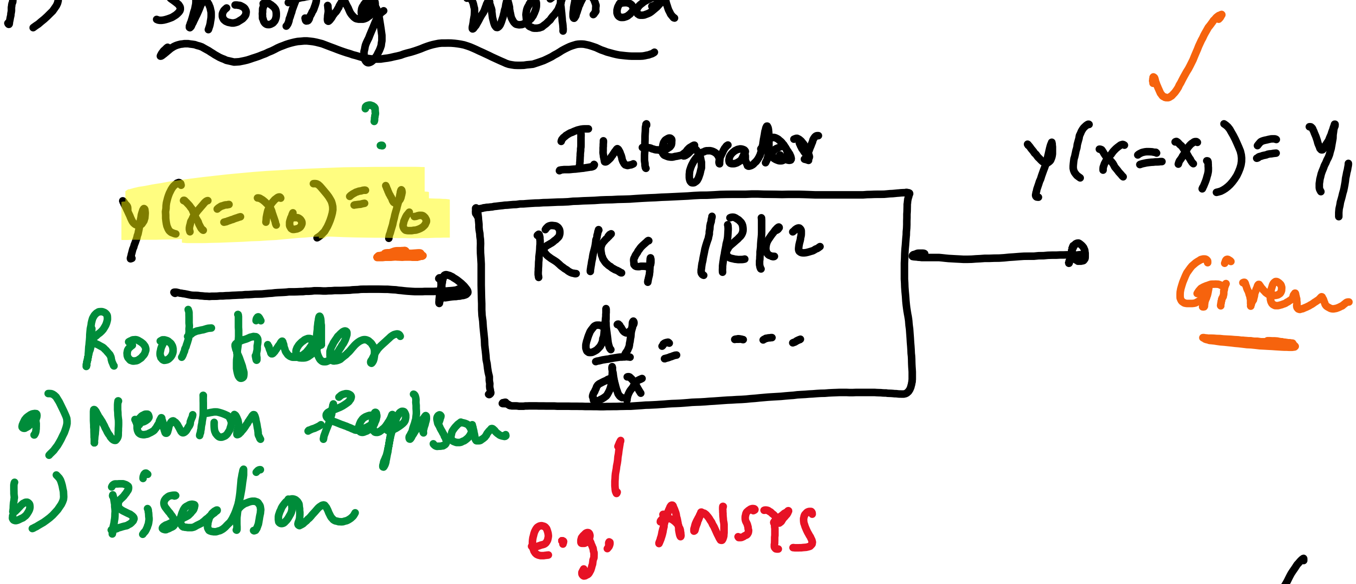
## (i) Shooting method




$$\frac{dy}{dx} = f(x,y) \quad ; \quad \text{Given } y(x=x_1)=y_1$$

Compute $y(x=x_0) = ?$

## EXAMPLE

Consider the ODE: $\frac{dy}{dx} = f(x,y) = -xy$

Given $y(x=0.5) = 1$ ; Compute $y(x=0) = y_0$

Use shooting method with Euler's integration and step size of $\Delta x = 0.1$

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$x = x_0 + i\Delta x$ ; $x_0 = 0$, $\Delta x = 0.1$

$x = 0, 0.1, 0.2, 0.3, 0.4, \boxed{0.5}$ $\quad i = 0, 1, \ldots, 5$

$y = y_0, ?, ?, ?, ?, \boxed{1}$

Euler's integration:

$$y_{i+1} = y_i + \Delta x \, f(x,y)$$

$$y_{i+1} = y_i + \Delta x \, (-x_i y_i)$$

$$y_{i+1} = (1 - \Delta x \, x_i) \, y_i$$

$$y_{i+1} = (1 - \Delta x \, x_i)\, y_i$$

$i=0$ ; $x_i = x_0 = 0$ ; $y_i = y_0$ ; $\Delta x = 0.1$

$$y_1 = \big(1 - (0.1)(0)\big) y_0 \Rightarrow y_1 = y_0 \quad -1$$

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$i=1$ ; $x_1 = x_0 + (1)\Delta x = 0.1$

$$y_2 = \big(1 - (0.1)(0.1)\big) y_1$$

$$y_2 = (0.99)\, y_1 = 0.99\, y_0 \quad \text{From } 1$$

2

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$i=2$ ; $x_2 = x_0 + 2\Delta x = 0.2$

$$y_3 = \big(1 - (0.1)(0.2)\big) y_2 = 0.98\, y_2$$

$$= 0.98\,(0.99)\, y_0 \quad \text{From } 2$$

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$i=3$ $\quad y_4 = \big(1 - 0.1\,(0.3)\big) y_2$

$$= (0.97)(0.98)(0.99)\, y_0$$

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$i=4$ $\quad y_5 = \big(1 - (0.1)(0.4)\big) y_4$

$$= (0.96)(0.97)(0.98)(0.99)\, y_0$$

$$y_5 = 0.9035\, y_0$$

$$y_5 = 0.9035\, y_0$$

$$x_5 = x_0 + i\Delta x = 0 + 5\,(0.1) = 0.5$$

$$y_5 = y(x_5) = y(0.5) = 0.9035\, y_0 = 1$$

$$y_0 = \frac{1}{0.9035} = 1.1068$$

$$\boxed{y_0 = 1.1068}$$

## (ii) Finite Difference

a) Rewrite the ODE using finite difference

b) Assume unknowns at grid points.

( e.g. $\frac{dy}{dx} = \frac{y_{i+1} - y_i}{\Delta x}$ ; $y_i, y_{i+1}$ are unknowns )

c) Set up equations based on boundary conditions

d) Solve for unknowns

could involve root finding

a) Newton Raphson

b) Secant.

## EXAMPLE

Consider the ODE: $\frac{dy}{dx} = f(x,y) = -xy$

Given $y(x=0.5) = 1$ ; Compute $y(x=0) = y_0$

Use finite difference with forward difference and step size of $\Delta x = 0.1$

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$x = 0, \; 0.1, \; 0.2, \; 0.3, \; 0.4, \; 0.5$

$y = y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5 = 1$

$\underbrace{y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5}_{\text{unknowns.}}$

Step b

$$\frac{dy}{dx} = \frac{y_{i+1} - y_i}{\Delta x} = -x_i y_i$$

$$y_{i+1} = (1 - \Delta x \, x_i) \, y_i$$

Step a

$$y_{i+1} = (1 - \Delta x \, x_i) y_i$$

$$i=0 \quad y_1 = [1-(0.1)(0)] y_0 = y_0$$

$$i=1 \quad y_2 = [1-(0.1)(0.1)] y_1 = 0.99 y_1$$

$$i=2 \quad y_3 = [1-(0.1)(0.2)] y_2 = 0.98 y_2$$

$$i=3 \quad y_4 = [1-(0.1)(0.3)] y_3 = 0.97 y_3$$

$$i=4 \quad y_5 = 1 = [1-(0.1)(0.4)] y_4 = 0.96 y_4$$

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$$-y_0 + y_1 = 0$$

$$-0.99 y_1 + y_2 = 0$$

$$-0.98 y_2 + y_3 = 0$$

$$-0.97 y_3 + y_4 = 0$$

$$0.96 y_4 = 1$$

$$-y_0 + y_1 = 0$$

$$-0.99y_1 + y_2 = 0$$

$$-0.98y_2 + y_3 = 0$$

$$-0.97y_3 + y_4 = 0$$

$$-0.96y_4 = 1$$

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$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -0.99 & 1 & 0 & 0 \\ 0 & 0 & -0.98 & 1 & 0 \\ 0 & 0 & 0 & -0.97 & 1 \\ 0 & 0 & 0 & 0 & -0.96 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A \quad y \quad = b$$

$$y = A \backslash b$$

ANSWER

$\boxed{y_0 = 1.1068;}$ $y_1 = 1.1068;$ $y_2 = 1.0958;$

$y_3 = 1.0739;$ $y_4 = 1.0417$