② Romberg Integration

Earlier (in the Richardson extrapolation)

Total error (trapezoidal) $\propto \Delta x^2$

But the total error

$$I = I_1 + c_1 \Delta x^2 + c_2 \Delta x^4 + c_3 \Delta x^6 + \dots$$

For Richardson extrapolation

$$\left.\begin{aligned} I &= I_1 + c_1 \Delta x_1^2 \\ I &= I_2 + c_1 \Delta x_2^2 \end{aligned}\right\} \quad I = \frac{I_2 r^2 - I_1}{r^2 - 1}$$

total error in I is $\propto \Delta x^4$

$$I_{1,n} = \frac{r^2 I_{0,n} - I_{0,n/2}}{r^2 - 1}$$

$\boxed{\Delta x^4}$

$$I_{2,n} = \frac{r^4 I_{1,n} - I_{1,n/2}}{r^4 - 1}$$

$\boxed{\Delta x^6}$

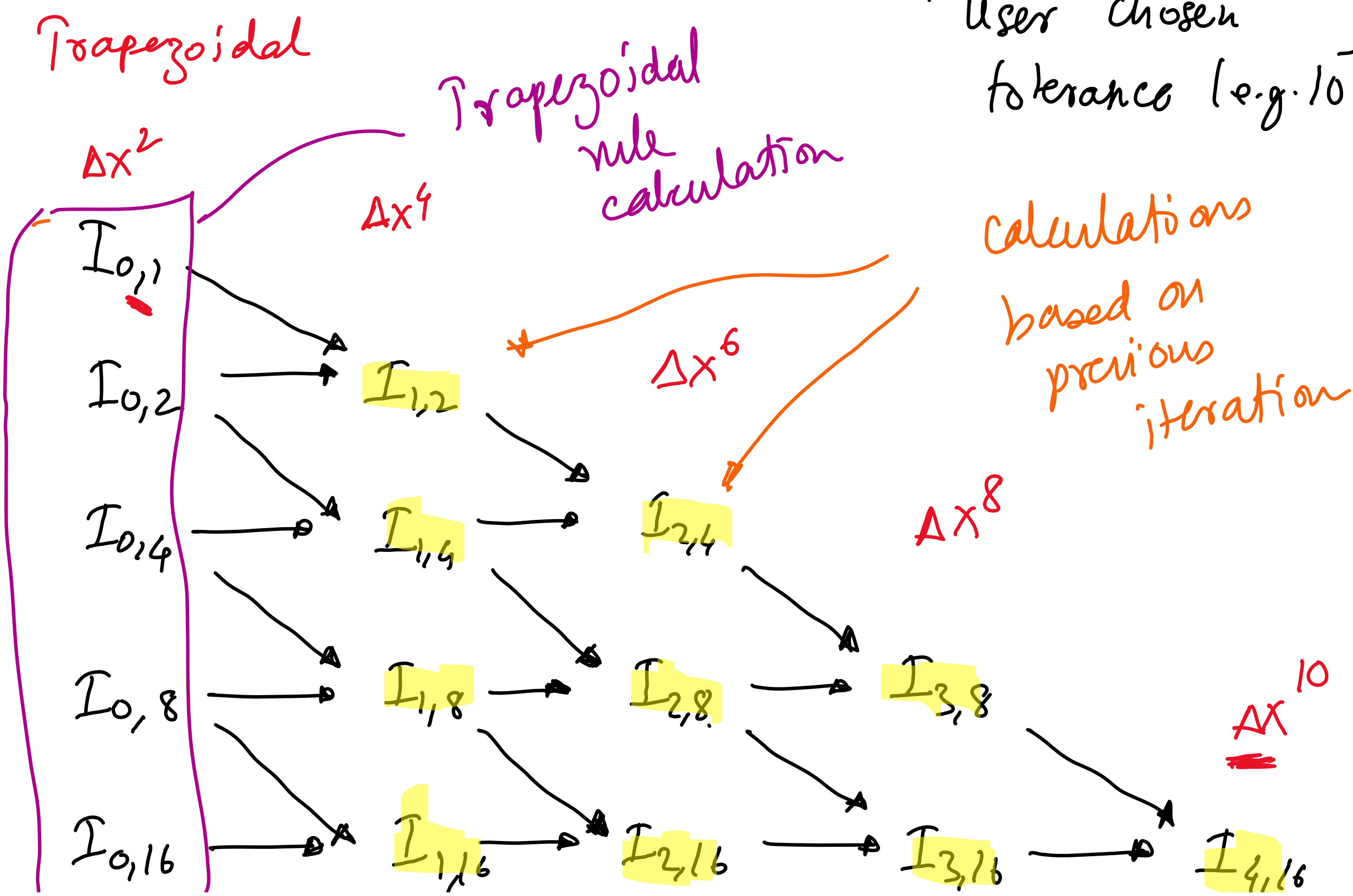
We can keep repeating this calculation to derive a general expression

$$I_{k,n} = \frac{r^{2k} I_{k-1,n} - I_{k-1,n/2}}{r^{2k} - 1} \checkmark$$

This calculation is done iteratively till

$$\left| I_{k-1,n} - I_{k,n} \right| < \varepsilon$$

$\uparrow$ User chosen tolerance (e.g. 10^{-4})



$\boxed{I_{0,16}} \longrightarrow I_{1,16} \longrightarrow I_{2,16} \longrightarrow I_{3,16} \longrightarrow I_{4,16}$

Compute $\int_0^1 \frac{4\,dx}{1+x^2}$

① Use the function quad in Octave/MATLAB to a tolerance 10^{-8}

② Romberg integration. For iteration 0, use trapezoidal rule starting at $n=2$

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① $f = @(x)\ 4./(1+x^2)$

$I_{exact} = \text{quad}(f, a, b, 1e^{-8}) = \pi = 3.14159265$

$$I = 4\int_0^1 \frac{dx}{1+x^2} = 4 \tan^{-1}(x)\Big|_0^1 = 4\left(\left(\frac{\pi}{4}\right) - 0\right)$$

$$= \pi$$

Iteration 0:

$n=2$ $\quad I_{0,1} = 3.1$

$n=4$ $\quad I_{0,2} = 3.13117647$

$n=8$ $\quad I_{0,4} = 3.13898849$

$n=16$ $\quad I_{0,8} = 3.14094161$

$n=32$ $\quad I_{0,16} = \underline{3.14}142989$

Trapezoidal rule in Octave

Iteration 1

$$I_{1,2} = \frac{2^2 I_{0,2} - I_{0,1}}{2^2-1} = \underline{3.1415}6863$$

$$I_{1,4} = \frac{2^2 I_{0,4} - I_{0,2}}{2^2-1} = 3.\underline{1415}9250$$

$$I_{1,8} = \frac{2^2 I_{0,8} - I_{0,4}}{2^2-1} = 3.\underline{1415}9265$$

$$I_{1,16} = \frac{2^2 I_{0,16} - I_{0,8}}{2^2-1} = \underline{3.14159265}$$

Iteration 2

$$I_{2,4} = \frac{2^4 I_{1,4} - I_{1,2}}{2^4-1} = 3.14159407$$

$$I_{2,8} = \frac{2^4 I_{1,8} - I_{1,4}}{2^4-1} = 3.14159266$$

$$I_{2,16} = \frac{2^4 I_{1,16} - I_{1,8}}{2^4-1} = 3.14159265$$

$$|I_{1,16} - I_{2,16}| < 10^{-8}$$

$$\boxed{I_{approx} = 3.14159265}$$