Higher order accuracy methods

① Richardson extrapolation

Trapezoidal rule, truncation error (TE) $\propto \Delta x^3$

Total error $= \int_a^b (\text{Truncation error})\, dx$

$$\propto \sum_{i=1}^{n} \Delta x^3$$

$$\propto n \, \Delta x^3$$

$$\propto \frac{(b-a)}{\cancel{\Delta x}} \Delta x^{\cancel{3}\,2}$$

$\Delta x = \dfrac{b-a}{n}$

Total error $\propto \Delta x^2$

Trapezoidal rule

$$I = I_1 + \underbrace{c_1 \Delta x_1^2}_{\text{total error}} \quad — ①$$

exact (I) Trapezoidal rule (I_1)

$$I = I_2 + c_1 \Delta x_2^2 \quad — ②$$

① − ②

$$0 = (I_1 - I_2) + C_1 \; (\Delta x_1^2 - \Delta x_2^2)$$

$$C_1 = \frac{I_2 - I_1}{\Delta x_1^2 - \Delta x_2^2}$$

$$I = I_1 + \frac{I_2 - I_1}{(\Delta x_1^2 - \Delta x_2^2)} \Delta x_1^2$$

Simplify

$$I = \frac{I_2 \, \Delta x_1^2 - I_1 \, \Delta x_1^2}{\Delta x_1^2 - \Delta x_2^2}$$

$$\boxed{\underline{I} = \frac{r^2 I_2 - I_1}{r^2 - 1} \Rightarrow \quad r = \frac{\Delta x_1}{\Delta x_2}}$$

Choose $r = 2$ $\quad \underline{I} = (4 I_2 - I_1)/3$

It can be shown that I is accurate to $\underline{\Delta x^4}$ ✓

→ For simpson's rule

Truncation error $\propto \Delta x^5$

Total error $\propto \underline{\Delta x^4}$

$$I = I_1 + C_1 \Delta x_1^4$$

$$I = I_2 + C_1 \Delta x_2^4$$

Solve for C_1 and assume $\frac{\Delta x_1}{\Delta x_2} = r$

$$\boxed{\underline{I} = \frac{r^4 I_2 - I_1}{r^4 - 1}}$$

The resulting I is accurate to $\underline{\Delta x^6}$

EXAMPLE:

compute $I = \int_0^{\pi/2} \cos(x)\, dx$ using trapezoidal rule with a step size of

1. $n=3$
2. $n=6$
3. Use Richardson's extrapolation to get a better estimate of the integral
4. Compare against analytical solution

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1. $dx = \frac{b-a}{n} = \frac{\pi/2 - 0}{3} = 0.5236$; $\boxed{x = a + i\,dx \quad i = 0, 1, 2, 3}$

x	0	0.5236	1.0472	1.5708
$\cos(x)$	1	0.8660	0.5	0
	f_0	f_1	f_2	f_3

$$I_1 = \frac{dx}{2}\left[f_0 + 2f_1 + 2f_2 + f_3 \right]$$

$$\boxed{I_1 = 0.9770}$$

② $n=6$

$$dx = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = 0.2618$$

$$x = a + i\,dx \qquad i = 0, 1, 2, 3, 4, 5, 6$$

x	0	0.2618	0.5236	0.7854	1.0472	1.3090	1.5708
$\cos(x)$	1	0.9659	0.8660	0.7071	0.5	0.2558	0
	f_0	f_1	f_2	f_3	f_4	f_5	f_6

$$I_2 = \frac{dx}{2}\left[f_0 + 2f_1 + 2f_2 + 2f_3 + 2f_4 + 2f_5 + f_6\right]$$

$$\boxed{I_2 = 0.9943}$$

③ $$I = \frac{(r^2 I_2) - I_1}{r^2 - 1} \qquad r = \frac{\Delta x_1}{\Delta x_2} = \frac{(b-a)/3}{(b-a)/6} = 2$$

$$I = \frac{(2^2 I_2) - I_1}{2^2 - 1} = \frac{4(0.9943) - 0.9770}{4-1}$$

$$\boxed{I = 1.001}$$

④ $$I_{exact} = \int_0^{\pi/2} \cos(x)\,dx = \sin x \Big|_0^{\pi/2} = \sin(\pi/2) - 0 = 1$$

$$\boxed{I_{exact} = 1}$$