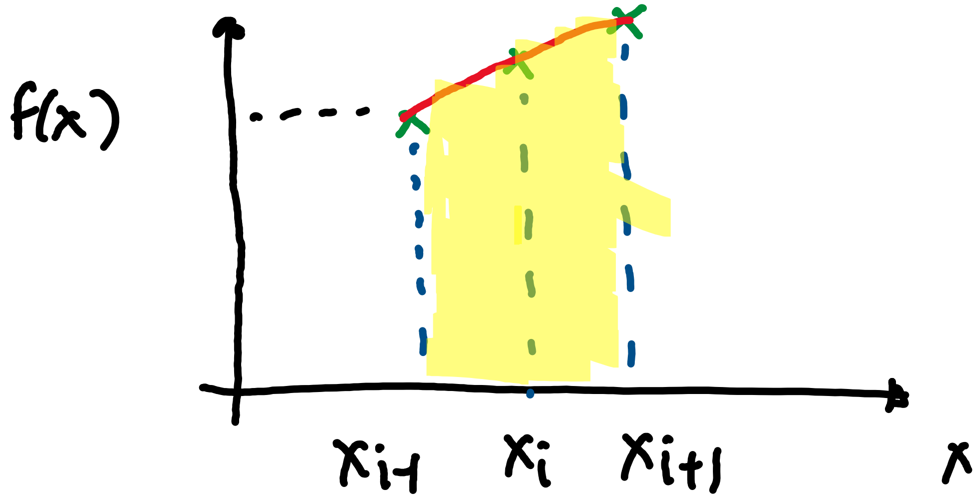
③ Simpsons 1/3 rule



We will approximate the curve using a quadratic polynomial

$$P_2(x) = A(x-x_i)^2 + B(x-x_i) + C \quad \text{— 3 unknowns}$$

3 eqns

$$\begin{cases} P_2(x_i) = C = f(x_i) = f_i \\ P_2(x_{i-1}) = A(x_{i-1}-x_i)^2 + B(x_{i-1}-x_i) + C = f_{i-1} \\ P_2(x_{i+1}) = A(x_{i+1}-x_i) + B(x_{i+1}-x_i) + C = f_{i+1} \end{cases}$$

Solving the system of 3 eq^s / 3 unknowns

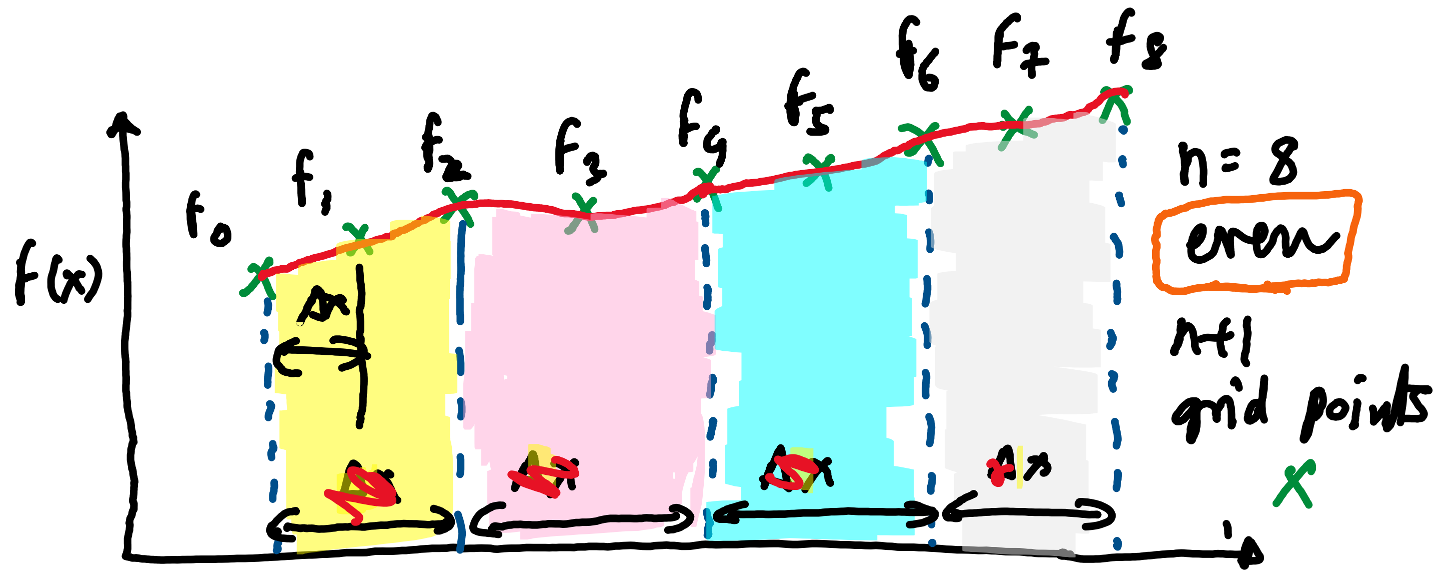
$$A = \frac{f_{i+1} - 2f_i + f_{i-1}}{2(\Delta x)^2} \qquad B = \frac{f_{i+1} - f_i}{\Delta x} \; ; \; C = f_i$$

$$\Delta x = x_{i+1} - x_i = x_i - x_{i-1}$$

$$I_i = \int_{x_{i-1}}^{x_{i+1}} \left[A(x-x_i)^2 + B(x-x_i) + C \right]$$

$$\boxed{\begin{aligned} I_i &= \frac{1}{3}\left[F(x_{i-1}) + 4F(x_i) + F(x_{i+1}) \right] \Delta x \\ I_i &= \frac{1}{3}\left[f_{i-1} + 4f_i + f_{i+1} \right] \Delta x \end{aligned}}$$

$\frac{1}{3}$ rule

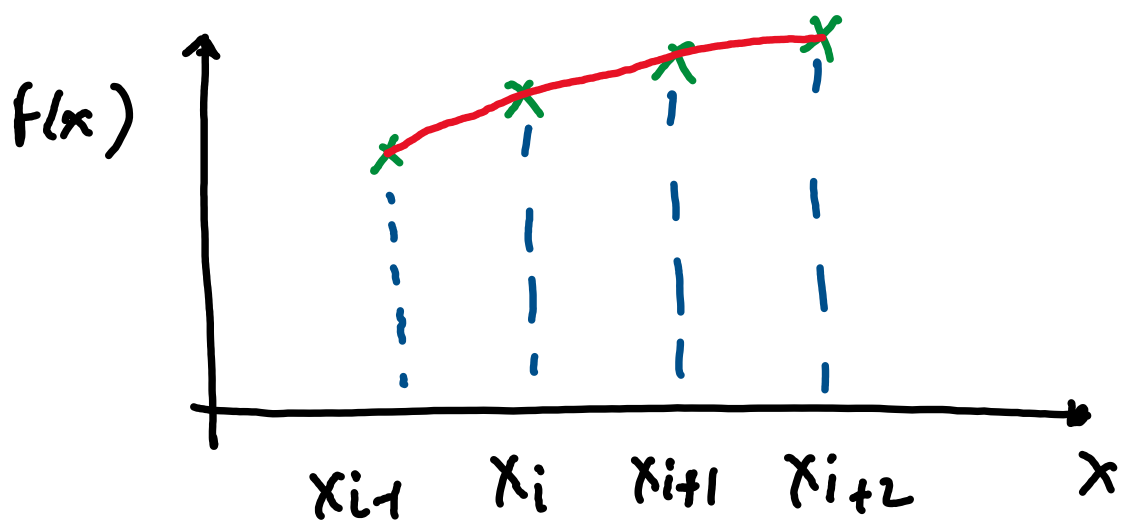


$$I = \Delta x \left\{ \frac{1}{3}\left[f_0 + 4f_1 + f_2\right] + \frac{1}{3}\left[f_2 + 4f_3 + f_4\right] + \frac{1}{3}\left[f_4 + 4f_5 + f_6\right] + \frac{1}{3}\left[f_6 + 4f_7 + f_8\right] \right\}$$

$$I = \frac{\Delta x}{3}\left\{ (f_0 + f_8) + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) \right\}$$

$$\boxed{I = \frac{\Delta x}{3}\left\{ (f_0 + f_n) + 4\sum_{i=1,3,5}^{n-1} f_i + 2\sum_{i=2,4,6}^{n-2} f_i \right\} \quad \boxed{n = \text{even}}}$$

④ Simpson's 3/8 th rule



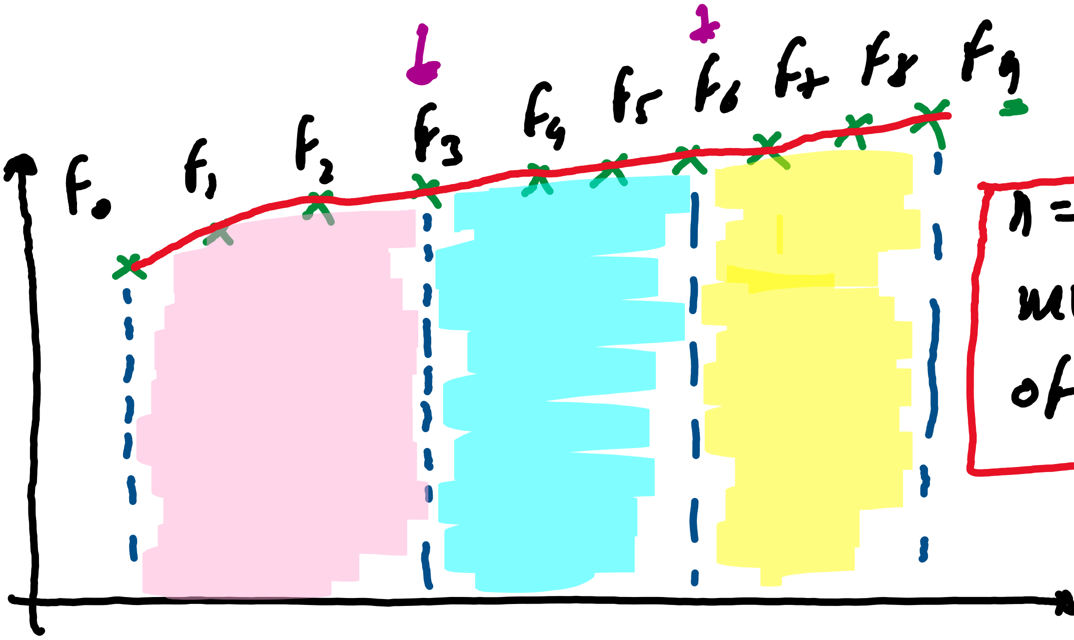
$$P_3(x) = A(x-x_i)^3 + B(x-x_i)^2 + C(x-x_i) + D$$

4 unknowns, 4 equations
(x_{i-1}, f_{i-1}) (x_i, f_i), (x_{i+1}, f_{i+1}), (x_{i+2}, f_{i+2})

Solve for A, B, C, D

$$I = \int_{x_{i-1}}^{x_{i+2}} \{A(x-x_i)^3 + B(x-x_i)^2 + C(x-x_i) + D\} dx$$

$$\boxed{I_i = \frac{3}{8} \{ f_{i-1} + 3f_i + 3f_{i+1} + f_{i+2} \} \Delta x}$$



$n = 3$
multiple
of 3

$$I = \frac{3}{8}\Delta x\Big[\{f_0+3f_1+3f_2+f_3\} + \{f_3+3f_4+3f_5+f_6\} + \{f_6+3f_7+3f_8+f_9\}\Big]$$

$$I = \frac{3}{8}\Delta x\Big[f_0+f_9+3\{f_1+f_2+f_4+f_5+f_7+f_8\} + 2\{f_3+f_6\}\Big]$$

$$\boxed{I = \frac{3}{8}\Delta x\Big[f_0+f_n+3\sum_{i=1,4,7}^{n-1}(f_i+f_{i+1})+2\sum_{i=3,6}^{n-3} f_i\Big]}$$

$n =$ multiple of 3

Example 2

Compute $I = \int_0^1 x^2 \, dx$ using

(a) Simpson's 1/3 rule

(b) Simpson's 3/8 rule

(c) Analytical calculation $= 0.3333$

Assume $n = 6$

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x	0	0.1667	0.3333	0.5	0.6667	0.8333	1
x^2	0	0.0278	0.1111	0.25	0.4444	0.6944	1
	f_0	f_1	f_2	f_3	f_4	f_5	f_6

(a) $I_{1/3} = \frac{1}{3} \Delta x \left[f_0 + f_6 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) \right]$

$$\boxed{I_{1/3} = 0.3337}$$

(b) $I_{3/8} = \frac{3}{8} \Delta x \left[f_0 + f_6 + 3(f_1 + f_2 + f_4 + f_5) + 2 f_3 \right]$

$$\boxed{I_{3/8} = 0.3333}$$