Numerical Integration

Compute $I = \int_a^b f(x)\,dx$

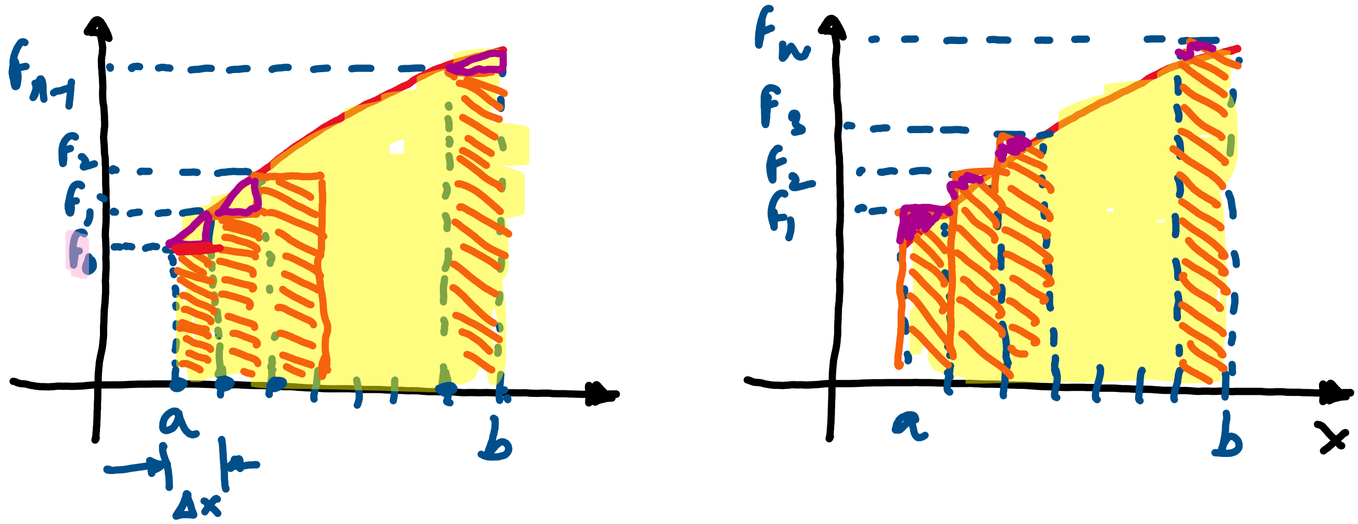
a, b are constants

Key idea:

Approxite $f(x)$ with a polynomial & then integrate the polynomial

4 methods	Polynomial
① Rectangular rule	constant
② Trapezoidal rule	linear
③ Simpsons $\frac{1}{3}$ rule	quadratic
④ Simpsons $3/8$ rule	cubic

① Rectangular rule



$$I = \int_a^b f(x)\,dx$$ = area under the curve $f(x)$ from a to b

I_{low} = shaded area

$= f_0 \Delta x + f_1 \Delta x + \ldots f_{n-1} \Delta x$

$= (f_0 + f_1 + \ldots f_{n-1}) \Delta x$

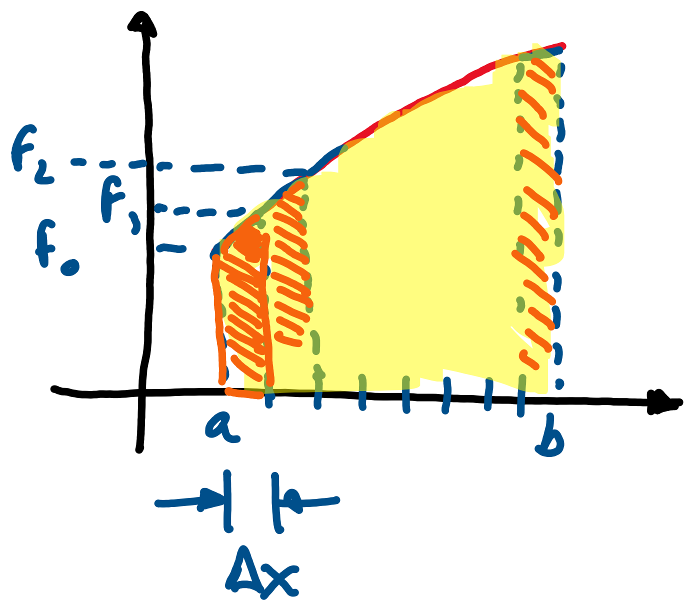
I_{upp} = shaded area

$= (f_1 + f_2 + \ldots f_n) \Delta x$

$$\boxed{I_{low} = \sum_{i=0}^{n-1} f_i \Delta x \qquad I_{upp} = \sum_{i=1}^{n} f_i \Delta x}$$

$$\boxed{\Delta x = \frac{b-a}{n}}$$

② Trapezoidal rule



$$I = \Delta x \left(\frac{f_0+f_1}{2}\right) + \Delta x \left(\frac{f_1+f_2}{2}\right) + \dots \Delta x \left(\frac{f_{n-1}+f_n}{2}\right)$$

$$I = \left(f_0 + 2f_1 + 2f_2 + \dots 2f_{n-1} + f_n\right)\frac{\Delta x}{2}$$

$$\boxed{I = \frac{\Delta x}{2}\left(f_0 + f_n + 2\sum_{i=1}^{n-1} f_i\right)}$$

$$\boxed{\Delta x = \frac{b-a}{n}}$$

Example 1

Compute $I = \int_0^1 x^2 \, dx$ using

(a) Rectangular rule

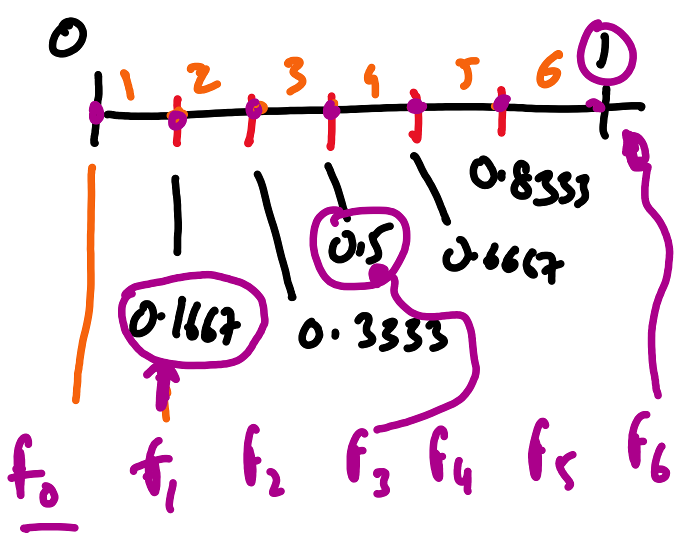
(b) Trapezoidal rule

(c) Analytical calculation

Assume $n = 6$

$n = 6$; $a = 0$; $b = 1$

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{6} = 0.1667$$



x	0	0.1667	0.3333	0.5	0.6667	0.8333	1
x^2	0	0.0278	0.1111	0.25	0.4444	0.6944	1
	f_0	f_1	f_2	f_3	f_4	f_5	f_6

(a) $$I_{low} = \sum_{i=0}^{5} f_i \Delta x$$

$$= (f_0 + f_1 + f_2 + f_3 + f_4 + f_5)\Delta x$$

$$= 0.2546$$

$$I_{upp} = \sum_{i=1}^{6} f_i \Delta x$$

$$= (f_1 + f_2 + f_3 + f_4 + f_5 + f_6)\Delta x$$

$$= 0.4213$$

(b) $$I_{Trap} = \frac{\Delta x}{2}\left[f_0 + f_6 + 2\sum_{i=1}^{5} f_i\right]$$

$$= \frac{\Delta x}{2}\left[f_0 + f_6 + 2(f_1 + f_2 + f_3 + f_4 + f_5)\right]$$

$$= 0.3380$$

(c) $$I = \int_0^1 x^2 dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3} - 0 = 0.3333$$

True