③ Newton's Divided Difference

Fit a curve through $(n+1)$ data points
$[x_0, f(x_0)], [x_1, f(x_1)], \ldots [x_n, f(x_n)]$

We are going to build intuition for the formula starting with 2 points to fit, then three, ... generalize to $n+1$ points.

Data for 2 points : $[x_0, f(x_0)], [x_1, f(x_1)]$

Assume: $f(x) = c_0 + c_1 (x - x_0)$

$\Rightarrow f(x_0) = c_0 + c_1 \underbrace{(x_0 - x_0)}_{0} \Rightarrow c_0 = f(x_0)$

$\Rightarrow f(x_1) = c_0 + c_1 (x_1 - x_0)$

$f(x_1) = f(x_0) + c_1 (x_1 - x_0)$

Solve for $c_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$

$$f(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$\boxed{f(x) = f(x_0) + F(x_1, x_0)(x - x_0)}$$

Data for 3 points : $[x_0, f(x_0)], [x_1, f(x_1)]$
$[x_2, f(x_2)]$

$$f(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1)$$

$$\Rightarrow f(x_0) = c_0 + c_1\cancel{(x_0 - x_0)} + c_2\cancel{(x_0 - x_0)}(x_0 - x_1)$$

$$\Rightarrow c_0 = f(x_0)$$

$$\Rightarrow f(x_1) = c_0 + c_1(x_1 - x_0) + c_2(x_1 - x_0)\cancel{(x_1 - x_1)}$$

$$\Rightarrow c_1 = \frac{f(x_1) - f(x_0)}{(x_1 - x_0)} = \underline{F(x_1, x_0)}$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0} + c_2 (x_2 - x_0)(x_2 - x_1)$$

Solve for c_2

$$c_2 = \frac{\left[\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)\right] - \left[\frac{f(x_1) - f(x_0)}{x_1 - x_0}\right]}{x_2 - x_0}$$

$$c_2 = \frac{F(x_2, x_1) - F(x_1, x_0)}{(x_2 - x_0)}$$

$$c_2 = F(x_2, x_1, x_0)$$

$$c_3 = F(x_3, x_2, x_1, x_0)$$

$$= \frac{F(x_3, x_2, x_1) - F(x_2, x_1, x_0)}{x_3 - x_0}$$

$$\boxed{a_n = F(x_n, x_{n-1}, \ldots x_1, x_0) = \frac{F(x_n, x_{n-1}, \ldots x_1) - F(x_{n-1}, x_{n-2}, \ldots x_0)}{x_n - x_0}}$$

$$f(x) = F(x_0) + (x - x_0) F(x_1, x_0) + \ldots$$

$$\ldots (x - x_0)(x - x_1) F(x_2, x_1, x_0) + \ldots$$

$$\vdots$$

$$\ldots (x - x_0)(x - x_1) \ldots (x - x_{n-1}) F(x_n, x_{n-1}, \ldots x_0)$$

Simplification for uniformly spaced points

$$x_i = x_0 + i\,\Delta x \quad ; \quad i = 1, 2, 3, \ldots n$$

$$c_0 = f(x_0)$$

$$c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f_0}{\Delta x}$$

$$c_2 = \frac{F(x_2, x_1) - F(x_1, x_0)}{x_2 - x_0} = \frac{f(x_2) - 2f(x_1) + f(x_0)}{2(\Delta x^2)}$$

$$= \frac{\Delta^2 f(x_0)}{2!\,(\Delta x)^2}$$

$$\vdots$$

$$c_n = F(x_n, x_{n-1}, \ldots x_1, x_0) = \frac{\Delta^n f_0}{n!\,(\Delta x)^n}$$

Newton-Gregory Forward Interpolation

$$f(x) = f(x_0) + \frac{\Delta f_0}{\Delta x}(x-x_0) + \frac{\Delta^2 f_0}{2!\,\Delta x^2}(x-x_0)(x-x_0-\Delta x)$$

$$\vdots$$

$$+ \ldots \frac{\Delta^n f_0}{n!\,(\Delta x)^n}(x-x_0)(x-x_0-\Delta x)\ldots$$

Newton-Gregory Backward Interpolation

$$f(x) = f(x_n) + \frac{\nabla f_n}{\Delta x}(x-x_n) + \frac{\nabla^2 f_0}{2!\,\Delta x^2}(x-x_n)(x-x_n-\Delta x)$$

$$\vdots$$

$$+ \ldots \frac{\nabla^n f_0}{n!\,(\Delta x)^n}(x-x_n)(x-x_n-\Delta x)\ldots$$

EXAMPLE

Given the following data

x	0	1	2	3
y	2	0	4	20

a) Obtain a fit using Newton-Gregory Forward interpolation

b) Obtain a fit using Newton-Gregory Backward interpolation

c) Estimate the value of y at x=0.5

(a)

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
$\boxed{0}$	2 (f_0)	$0-2=-2$ (Δf_0)	$4-(-2)=6$ ($\Delta^2 f_0$)	$12-6=6$ ($\Delta^3 f_0$)
$\boxed{1}$	0	$4-0=4$	$16-4=12$	
2	4	$20-4=16$		
3	20			

$$f(x) = 2 + \frac{(-2)}{\underset{\Delta x=1}{}}(x-0) + \frac{6}{2!\,(\Delta x)^2_{=1^2}}(x-0)(x-1) + \ldots$$

$$\frac{6\,(x-0)(x-1)(x-2)}{3!\,(\Delta x)^3_{=1^3}}$$

$$f(x) = 2 - 2x + 3x(x-1) + x(x-1)(x-2)$$

$$f(x) = 2 - 2x + 3x^2 - 3x + x^3 - 3x^2 + 2x$$

$$\boxed{f(x) = x^3 - 3x + 2} \quad *$$

(b)

x	f	Δf	$\Delta^2 f$	$\Delta^3 f$
0	2			
1	0	$0-2=-2$		
2	4	$4-0=4$	$4-(-2)=6$	
3	$\boxed{20}$	$20-4=\boxed{16}$	$16-4=\boxed{12}$	$12-6=\boxed{6}$

$$f(x) = 20 + \frac{16}{\underbrace{\Delta x}_{=1}}(x-3) + \frac{12}{2!\,\underbrace{(\Delta x^2)}_{=1^2}}(x-3)(x-2) + \dots$$

$$+ \frac{6}{3!\,\underbrace{\Delta x^3}_{=1}}(x-3)(x-2)(x-1)$$

$$f(x) = 20 + 16(x-3) + 6(x-3)(x-2) + \dots$$

$$+ 1(x-3)(x-2)(x-1)$$

$$\boxed{f(x) = x^3 - 3x + 2}$$ *

(c) $f(0.5) = 0.5^3 - 3(0.5) + 2$

$$\boxed{f(0.5) = 0.625}$$ ANSWER