③ Matrix inverse using Gauss-Jordan elimination

If we have a square matrix A then inverse A^{-1} is such that

$$AA^{-1} = A^{-1}A = I$$

How to compute A^{-1} with Gauss-Jordan elimination

$$[\, A \;\vdots\; I \,] \xrightarrow{\text{Gauss-Jordan}} [\, I \;\vdots\; A^{-1} \,]$$

If $Ax = b$

then $x = A^{-1} b$

EXAMPLE: Solve $Ax = b$ using matrix inverse obtained from Gauss-Jordan

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -3 & 1 \\ 2 & 2 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$$

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① Goal is to compute A^{-1} ② $x = A^{-1}b$

$$[A \,\vdots\, I]$$

$$\left[\begin{array}{ccc:ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 = R_1/2$

$$\left[\begin{array}{ccc:ccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 = R_2 - \frac{1}{1} R_1$; $R_3 = R_3 - \frac{2}{1} R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0.5 & 0.5 & 0.5 & 0 & 0 \\ 0 & \underline{-3.5} & 0.5 & -0.5 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$R_2 = R_2 / -3.5$$

$$R_1 = R_1 - 0.5 R_2 \; ; \; R_3 = R_3 - R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0.5715 & 0.4285 & 0.1429 & 0 \\ 0 & 1 & -0.1429 & 0.1429 & -0.2857 & 0 \\ 0 & 0 & -1.8571 & -1.1429 & 0.2857 & 1 \end{array}\right]$$

$$R_3 = R_3 / (-1.8571)$$

$$R_1 = R_1 - 0.5715 R_3 \; ; \; R_2 = R_2 + 0.1429 R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.0767 & 0.2308 & 0.3077 \\ 0 & 1 & 0 & 0.2304 & -0.3076 & -0.076 \\ 0 & 0 & 1 & 0.6154 & -0.1538 & -0.5384 \end{array}\right]$$

$$\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0.0767 & 0.2308 & 0.3077 \\ 0.2304 & -0.3076 & -0.076 \\ 0.6154 & -0.1538 & -0.5384 \end{bmatrix}$$

$$b = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$$

$$x = A^{-1} b = \begin{bmatrix} 0.0767 & 0.2308 & 0.3077 \\ 0.2304 & -0.3076 & -0.076 \\ 0.6154 & -0.1538 & -0.5384 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$$

$$x = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

do this calculation & check answer $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$