① Gauss-elimination

Simple example

$$a_{11} x_1 + a_{12} x_2 = b_1$$

$$a_{21} x_1 + a_{22} x_2 = b_2$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22}' \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \end{bmatrix}$$

① $a_{22}' x_2 = b_2'$ ✓

②

$$a_{11} x_1 + a_{12} x_2 = b_1'$$

$$a_{11} x_1 + a_{12} \left(\frac{b_2'}{a_{22}'} \right) = b_1'$$

Gauss-elimination simplifies $AX=b$ to

$$\begin{bmatrix} x & x & x & x & \cdots \\ 0 & x & x & \cdots & \\ 0 & 0 & x & \cdots & \\ 0 & 0 & 0 & x & \cdots \\ 0 & 0 & 0 & 0 & x \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2' \\ b_3' \\ \vdots \\ b_n' \end{bmatrix}$$

How do we do this?

Pivot

$$a_{11} x_1 + a_{12} x_2 = b_1 \qquad \text{— (I)}$$

$$a_{21} x_1 + a_{22} x_2 = b_2 \qquad \text{— (II)}$$

$$\text{(II)} - \text{(I)}\left(\frac{a_{21}}{a_{11}}\right)$$

$$(\cancel{a_{21} x_1} + a_{22} x_2) - \frac{a_{21}}{a_{11}}(\cancel{a_{11} x_1} + a_{12} x_2) = b_1 - \frac{a_{21}}{a_{11}} b_2$$

$$\underbrace{\left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right)}_{a_{22}'} x_2 = \underbrace{b_1 - \frac{a_{21}}{a_{11}} b_2}_{b_2'}$$

EXAMPLE:

Solve the following set of equations

$$\left.\begin{aligned} 2x_1 + x_2 + x_3 &= 7 \\ x_1 - 3x_2 + x_3 &= -2 \\ 2x_1 + 2x_2 - x_3 &= 3 \end{aligned}\right\}$$

blank

$x_1 = 1$

$x_2 = 2$

$x_3 = 3$

$$\begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \\ R_3 \rightarrow \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 7 \\ 1 & -3 & 1 & -2 \\ 2 & 2 & -1 & 3 \end{array}\right]$$

$[A \mid b]$

$$R_2 = R_2 - \frac{1}{2} R_1 \; ; \; R_3 = R_3 - \frac{2}{2} R_1$$

$$\begin{bmatrix} 2 & 1 & 1 & 7 \\ 1 - \frac{1}{2}(2) & -3 - \frac{1}{2}(1) & 1 - \frac{1}{2}(1) & -2 - \frac{1}{2}\, 7 \\ 2 - 2 & 2 - 1 & -1 - 1 & 3 - 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 7 \\ 1-\underline{\tfrac{1}{2}(2)} & -3-\tfrac{1}{2}(1) & 1-\tfrac{1}{2}(1) & -2-\tfrac{1}{2}\,7 \\ 2-2 & 2-1 & -1-1 & 3-7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 7 \\ 0 & -3.5 & 0.5 & -5.5 \\ 0 & 1 & -2 & -4 \end{bmatrix}$$

$$R_3 = R_3 - \frac{1}{-3.5} R_2$$

$$\begin{bmatrix} 2 & 1 & 1 & 7 \\ 0 & -3.5 & 0.5 & -5.5 \\ 0 & 1-\left(-\frac{1}{3.5}\right)(-3.5) & 2-\left(-\frac{1}{3.5}\right)(0.5) & -4-\left(-\frac{1}{3.5}\right)(-5.5) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 7 \\ 0 & -3.5 & 0.5 & -5.5 \\ 0 & -1+\left(-\frac{1}{3.5}\right)(-3.5) & 2-\left(-\frac{1}{3.5}\right)(0.5) & -4-\left(-\frac{1}{3.5}\right)(-5.5) \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 & 7 \\ 0 & -3.5 & 0.5 & -5.5 \\ 0 & 0 & -1.857 & -5.57 \end{bmatrix}$$

$$2x_1 + x_2 + x_3 = 7 \qquad -(i)$$

$$-3.5x_2 + 0.5x_3 = -5.5 \qquad -(ii)$$

$$-1.857x_3 = -5.57 \qquad -(iii)$$

From (iii) $x_3 = 3$

From (ii) $-3.5x_2 + 0.5(3) = -5.5$

$x_2 = 2$

From (i) $2x_1 + 2 + 3 = 7$

$x_1 = 1$