

Elliptic PDE

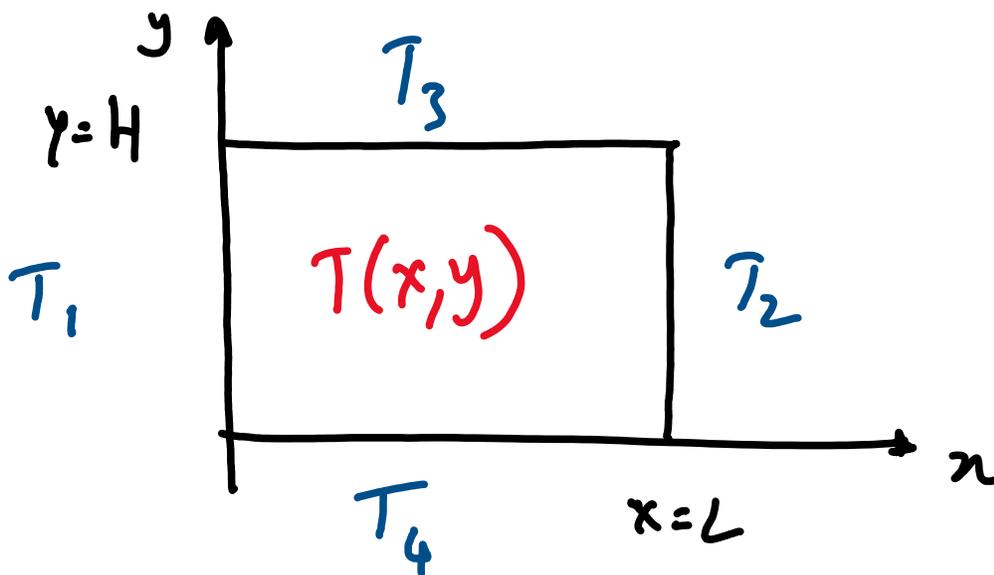
Laplace equation

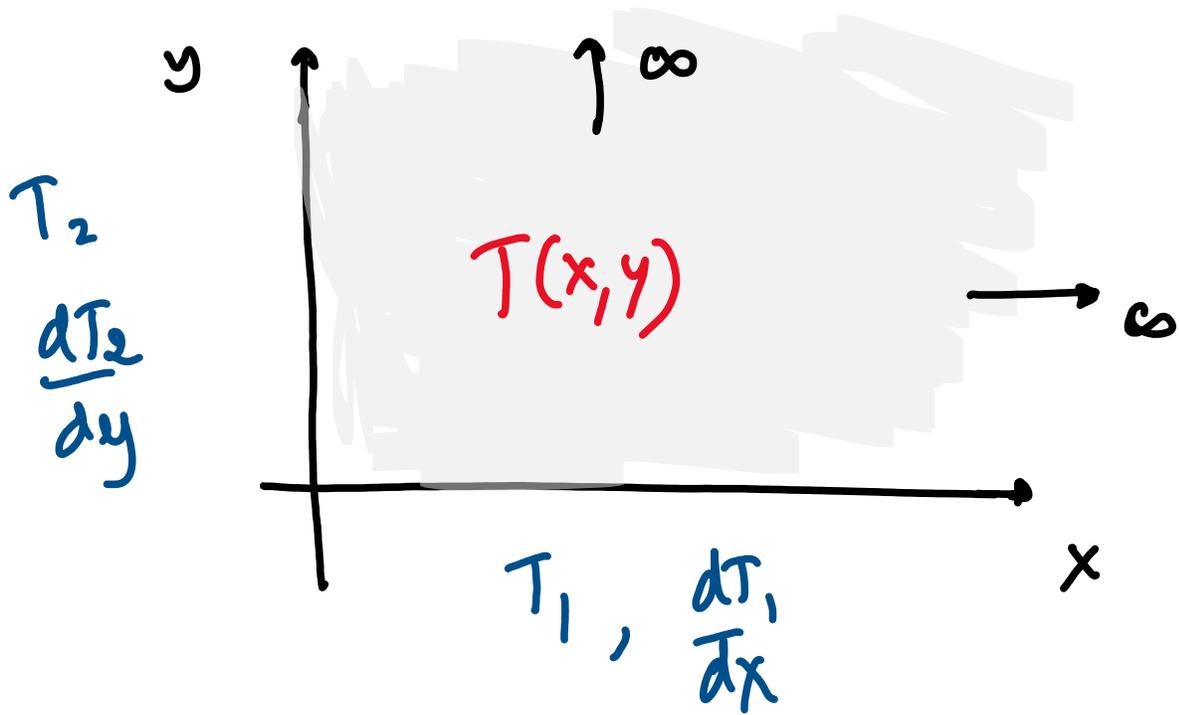
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

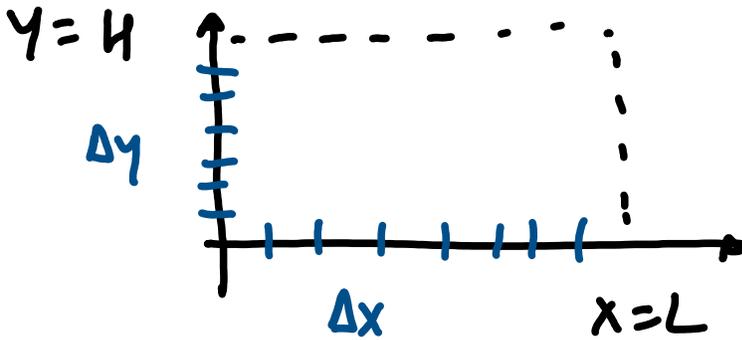
$$B = 0 ; A = C = 1 ;$$

$$B^2 - 4AC = 0 - 4 < 0$$

4 Boundary conditions; 2 for x , 2 for y .







$$\Delta x = \frac{L}{m} ; \quad \Delta y = \frac{H}{n} ; \quad m, n \text{ \# grid points}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} \quad i \equiv x\text{-pos}^n$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} \quad j \equiv y\text{-pos}^n$$

substitute these in

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = 0$$

$$\text{let } \Delta x = \Delta y$$

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0$$

Laplace eqⁿ

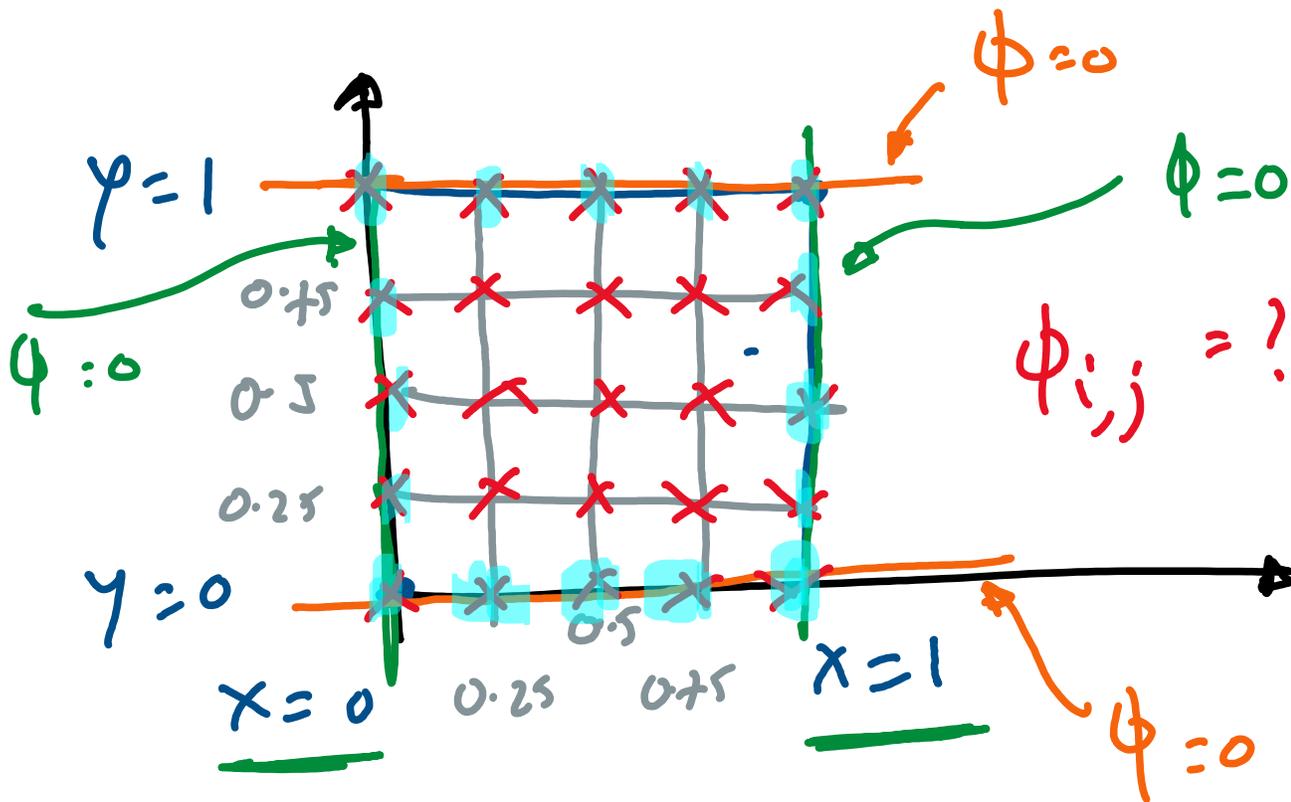
EXAMPLE: Solve for ϕ in the

Poisson's equation: $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -4$

B.C. $x=0, x=1$ $\phi=0$

$y=0, y=1$ $\phi=0$

Assume $\Delta x = \Delta y = 0.25$. Compute $\phi(x, y)$



$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = -4$$

$$\Delta x = \Delta y = 0.25$$

$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = -4\Delta x^2$$

$$= (-4)(0.25)^2$$

$$= \underline{\underline{-0.25}}$$

$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = -0.25$$

$i \equiv x\text{-position}$; $j \equiv y\text{-position}$

$$x = \Delta x i$$

$$= 0.25 i$$

i	0	1	2	3	4
x	0	0.25	0.5	0.75	1

$$y = \Delta y j$$

$$= 0.25 j$$

j	0	1	2	3	4
y	0	0.25	0.5	0.75	1

$i \backslash j$	0	1	2	3	4
$X \backslash Y$	0	0.25	0.5	0.75	1
0	0	0	0	0	0
1	0	0.1719	0.2188	0.1719	0
2	0	0.2188	0.2812	0.2188	0
3	0	0.1719	0.2188	0.1719	0
4	1	0	0	0	0



$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = -0.25$$

$$\underline{i=1} \quad \underline{j=1} \quad \phi_{2,1} + \cancel{\phi_{0,1}} + \phi_{1,2} + \cancel{\phi_{1,0}} - 4\phi_{1,1} = -0.25$$

$$\underline{i=1} \quad \underline{j=2} \quad \phi_{2,2} + \cancel{\phi_{0,2}} + \phi_{1,3} + \phi_{1,1} - 4\phi_{1,2} = -0.25$$

$$\underline{i=1} \quad \underline{j=3} \quad \phi_{2,3} + \cancel{\phi_{0,3}} + \cancel{\phi_{1,4}} + \phi_{1,2} - 4\phi_{1,3} = -0.25$$

$$i=2 \quad j=1 \quad \phi_{3,1} + \phi_{1,1} + \phi_{2,2} + \cancel{\phi_{2,0}} - 4\phi_{2,1} = -0.25$$

$$i=2 \quad j=2 \quad \phi_{3,2} + \phi_{1,2} + \phi_{2,3} + \phi_{2,1} - 4\phi_{2,2} = -0.25$$

$$\underline{i=2} \quad \underline{j=3} \quad \phi_{3,3} + \phi_{1,3} + \cancel{\phi_{2,4}} + \phi_{2,2} - 4\phi_{2,3} = -0.25$$

$$i=3 \quad j=1 \quad \cancel{\phi_{4,1}} + \phi_{2,1} + \phi_{3,2} + \cancel{\phi_{3,0}} - 4\phi_{3,1} = -0.25$$

$$i=3 \quad j=2 \quad \cancel{\phi_{4,2}} + \phi_{2,2} + \phi_{3,3} + \phi_{3,1} - 4\phi_{3,2} = -0.25$$

$$\underline{i=3} \quad \underline{j=3} \quad \cancel{\phi_{4,3}} + \phi_{2,3} + \cancel{\phi_{3,4}} + \phi_{3,2} - 4\phi_{3,3} = -0.25$$

$$\begin{bmatrix}
 -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
 \end{bmatrix}
 \begin{bmatrix}
 \phi_{11} \\
 \phi_{12} \\
 \phi_{13} \\
 \phi_{21} \\
 \phi_{22} \\
 \phi_{23} \\
 \phi_{31} \\
 \phi_{32} \\
 \phi_{33}
 \end{bmatrix}
 =
 \begin{bmatrix}
 -0.25 \\
 -0.25 \\
 -0.25 \\
 -0.25 \\
 -0.25 \\
 -0.25 \\
 -0.25 \\
 -0.25 \\
 -0.25
 \end{bmatrix}$$

$$A X = b$$

$$X = A \setminus b$$

$$\phi_{1,1} = \phi_{1,3} = \phi_{3,1} = \phi_{3,3} = 0.1719$$

$$\phi_{1,2} = \phi_{2,1} = \phi_{2,3} = \phi_{3,2} = 0.2188$$

$$\phi_{2,2} = 0.2812$$