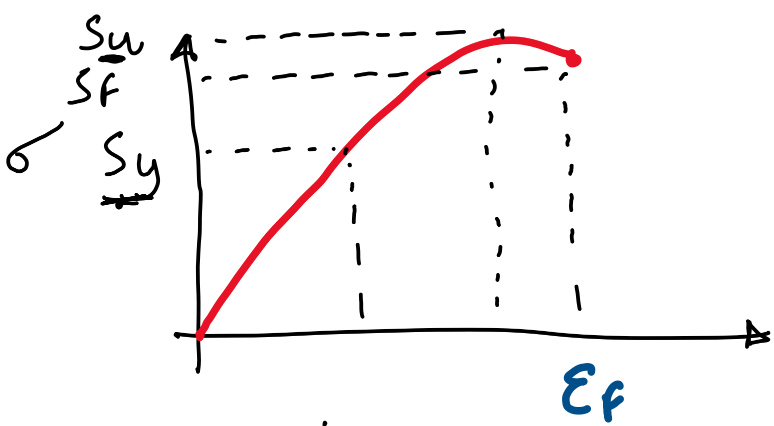
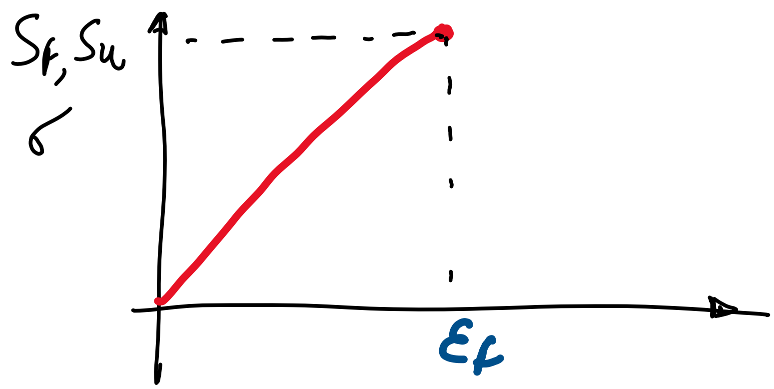
Failure theories

- Why materials fail?
- How can we predict failure?



Ductile



Brittle

- S_u, S_f are distinct
- Failure based on S_y

- S_f, S_u are the same
- Failure is based on S_f

Theories of Failure

Ductile

1. Maximum shear stress theory (MSS)
2. Distortion energy theory (DE)
3. Ductile Coulomb-Mohr Theory (DCM)

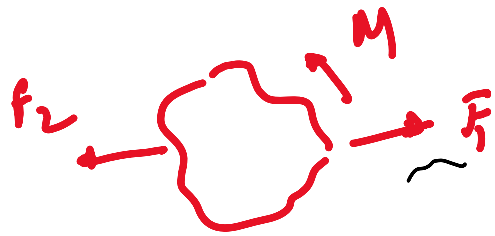
Brittle

1. Maximum Normal Stress theory (MNS)
2. Brittle Coulomb-Mohr theory (BCM)
3. Modified Mohr theory (MM)

① Maximum Shear Stress Theory

The material yields when the maximum shear stress equals or exceeds the maximum shear stress in a tension test specimen of the same material during yielding.

①

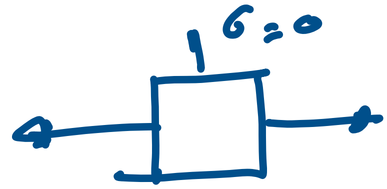


Compute max shear stress τ_{max} ✓

e.g. Mohr's circle.

② $S_y = \sigma$

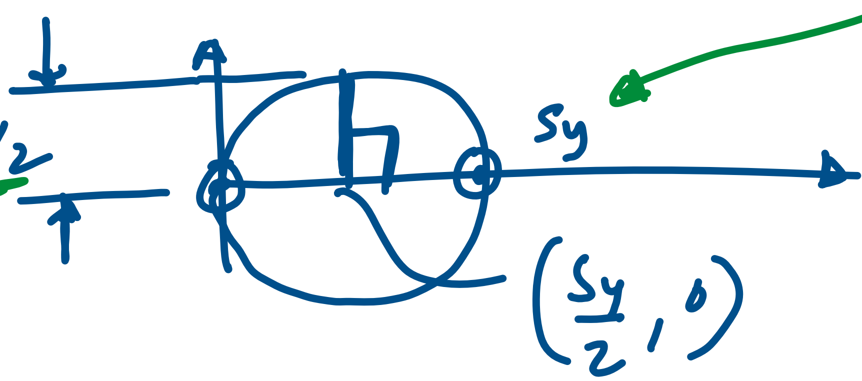
yields when $\sigma = S_y$



failure criteria

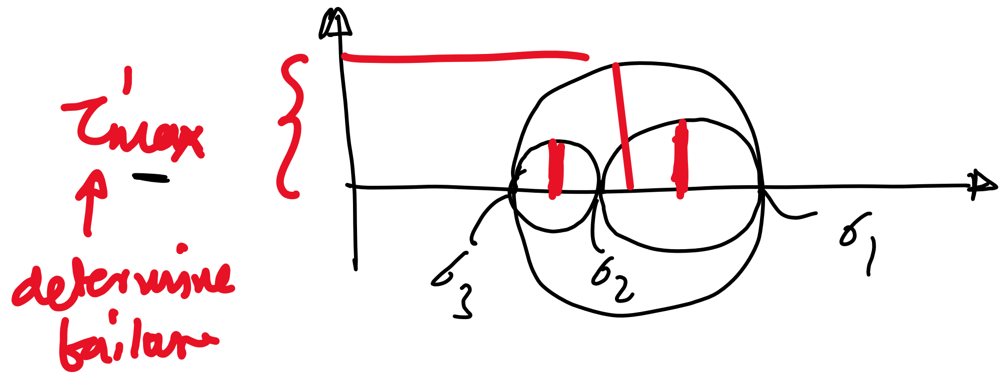
$$\tau_{max} \geq \frac{S_y}{2}$$

$\tau_{max} = S_y/2$



How to apply the theory in 3D

3 stresses : $\sigma_a, \sigma_b, \sigma_c$



Re-number $\sigma_1 > \sigma_2 > \sigma_3$

$$\tau'_{max} = \left(\frac{\sigma_1 - \sigma_3}{2}\right) \geq \frac{S_y}{2}$$

$$\sigma_1 - \sigma_3 \geq S_y$$

uniaxial case

How to use this for design

$$n_d / n = \frac{\text{max shear stress}}{\text{allowable stress}} = \frac{\tau_{max}}{\tau'_{max}}$$

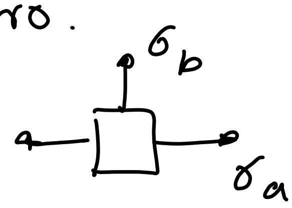
design factor/ factor of safety

$$= \frac{S_y/2}{(\sigma_1 - \sigma_3)/2}$$

$$= S_y/(\sigma_1 - \sigma_3)$$

$$= \frac{S_y}{2} / (\sigma_1 - \sigma_3)$$

For 2D case, one stress is alway zero.



Example $\sigma_a = 5$; $\sigma_B = -3$ MPa

$\sigma_c = 0$

Number the stresses σ_1, σ_2, σ_3

$$\sigma_1 > \sigma_2 > \sigma_3$$

$\sigma_1 = \sigma_A = 5$; $\sigma_2 = \sigma_c = 0$; $\sigma_3 = \sigma_b = -3$

$$5 > 0 > -3$$

$$n_d / n = \frac{S_y}{\sigma_1 - \sigma_3} = \frac{S_y}{8}$$

$\sigma_1 \to 5$, $\sigma_3 \to -3$

Example 2: $\sigma_A = 5$; $\sigma_B = 1$; $\sigma_C = 0$

$$\sigma_1 > \sigma_2 > \sigma_3 \Rightarrow 5 > 1 > 0$$

$5 = \sigma_1$, $0 = \sigma_3$

$$n_d / n = \frac{S_y}{\sigma_1 - \sigma_3} = \frac{S_y}{5}$$

$\sigma_1 \to 5$, $\sigma_3 \to 0$

2D case : $\sigma_a, \sigma_b, \sigma_c = 0$

① $\sigma_a \geq 0 \ ; \ \sigma_b \geq 0$

$\sigma_1 = \sigma_A$ or $\sigma_1 = \sigma_B$

$\sigma_3 = \sigma_c = 0$

Failure: $\tau_{max}' = S_y$

$\sigma_1 - \sigma_3 = S_y$

Either $\sigma_A = S_y$ or $\sigma_B = S_y$ } depends on if $\sigma_A > \sigma_B$ or $\sigma_B > \sigma_A$ — ①

② $\sigma_A \geq 0 \geq \sigma_B$

$\sigma_1 = \sigma_A \ ; \ \sigma_3 = \sigma_B$

$$\frac{x}{a} + \frac{y}{b} = 1$$

Failure $\sigma_1 - \sigma_3 = S_y$

$\sigma_A - \sigma_B = S_y$ — ②

③ $\sigma_B \geq 0 \geq \sigma_A$

$\sigma_B - \sigma_A = S_y$ — ③

④ $0 \geq \sigma_A \geq \sigma_B$, $0 \geq \sigma_B \geq \sigma_A$

$\sigma_1 = 0$

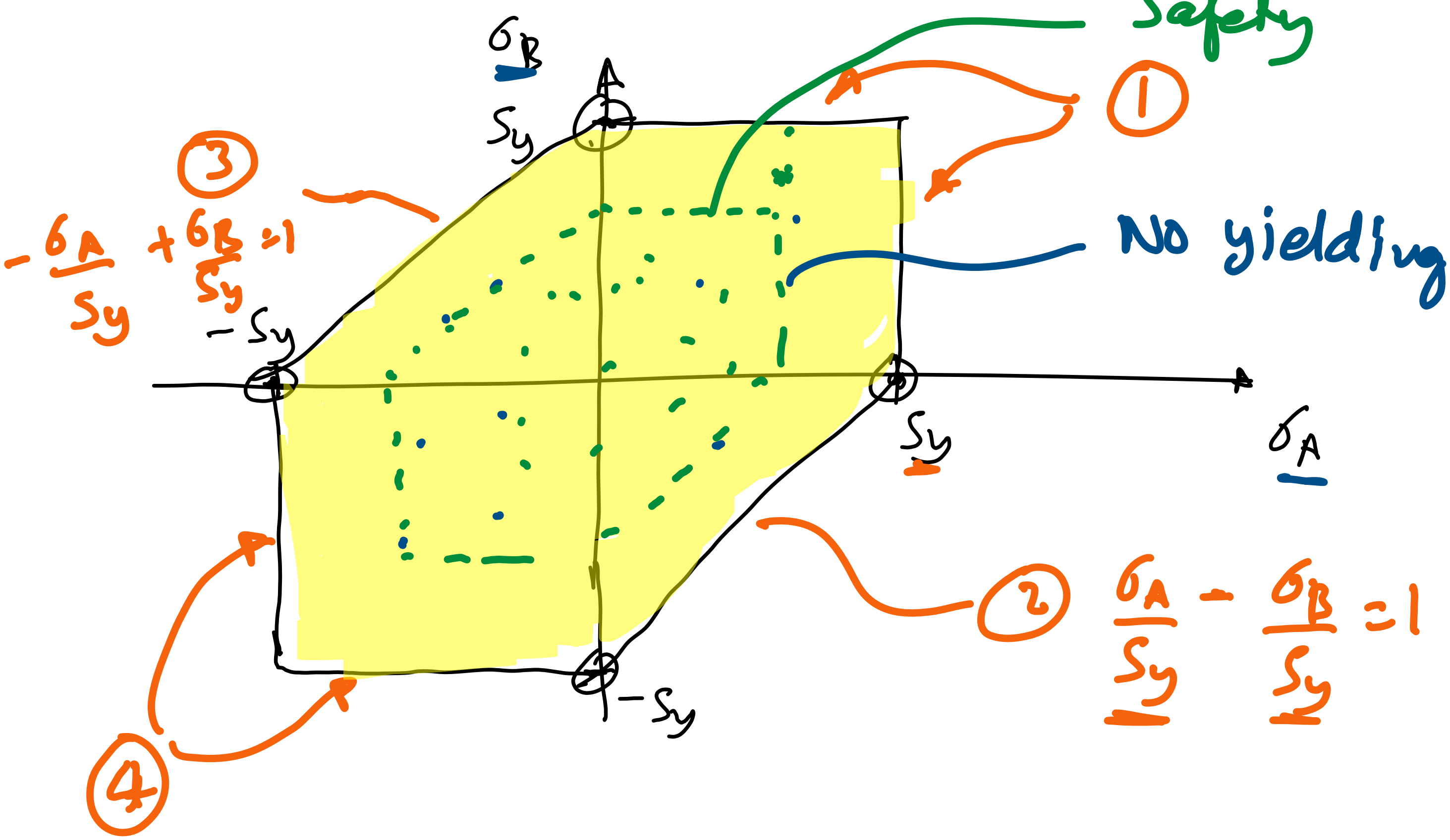
$\sigma_3 = \sigma_B$ or $\sigma_3 = \sigma_A$

failure $\sigma_1 - \sigma_3 = S_y$

$0 - \sigma_B = S_y$ or $0 - \sigma_A = S_y$

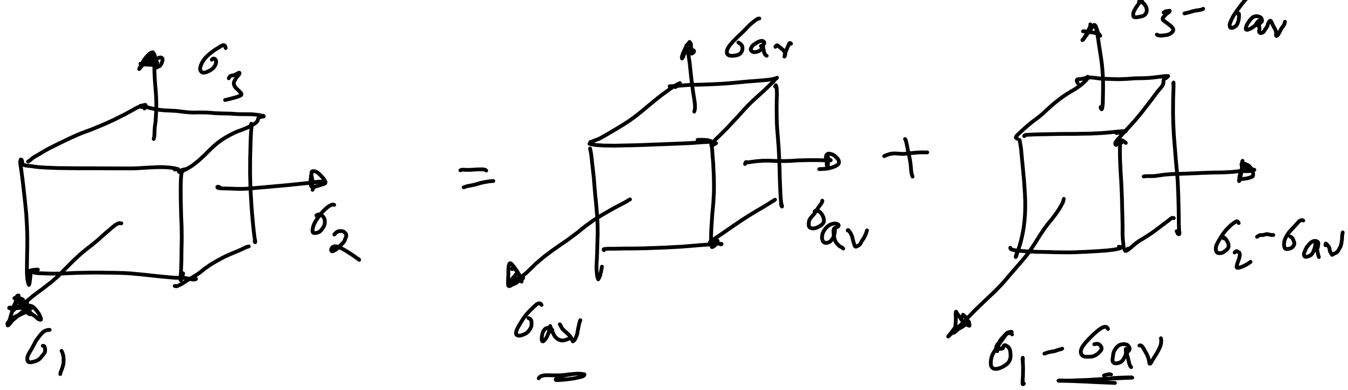
$\sigma_B = -S_y$, $\sigma_A = -S_y$ ④

Plotting Equations ① – ④



② Distortion energy theory

Yielding occurs when [distortion strain energy per unit volume] reaches or exceeds the [distortion energy per unit volume in a simple tension/compression experiment.] of the same material



General stress = Hydrostatic + Distortion
(a) (b) (c)

energy c?

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

energy a = energy b + energy c

energy c = energy a − energy b

energy in a $= \frac{1}{2}\left[\underline{\varepsilon_1}\sigma_1 + \underline{\varepsilon_2}\sigma_2 + \underline{\varepsilon_3}\sigma_3\right]$

But $\varepsilon_1 = \frac{1}{E}\left[\sigma_1 - \nu(\sigma_2 + \sigma_3)\right]$

and so on . . .

energy in (a) $= \frac{1}{2E}\left\{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right\}$ — (I)

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energy b $= ?$

$\sigma_{av} = \left(\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right)$

$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_{av}$ in (I)

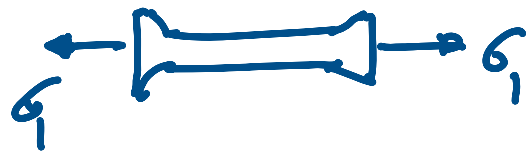
energy in b $= \frac{3(1-2\nu)}{E}\left[\frac{\sigma_1 + \sigma_2 + \sigma_3}{3}\right]^2$ — (II)

(c) energy in (c) = (I) − (II)

energy → DE

$$DE = \left(\frac{1+\nu}{3E}\right)\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right] \quad \text{— (III)}$$

simple tension/compression



→ $\sigma_1 = S_y$; $\sigma_2 = \sigma_3 = 0$

Substitute in (III) $DE = \left(\frac{1+\nu}{3E}\right)\left[\frac{(S_y-0)^2+(0)^2+(0-S_y)^2}{2}\right]$

$$DE = \left(\frac{1+\nu}{3E}\right) S_y^2 \quad \text{— (IV)}$$

DE failure $\sqrt{(III)} \geq \sqrt{(IV)}$

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{\frac{1}{2}} \geq S_y$$

σ' − von-Mises stresses.

DE failure $\sqrt{(III)} \geq \sqrt{(IV)}$

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{\frac{1}{2}} \geq S_y$$

σ' - von-Mises stresses.

Distortion energy (DE) for 2D case

$\sigma_1 = \sigma_A$; $\sigma_2 = \sigma_B$; $\sigma_3 = 0$

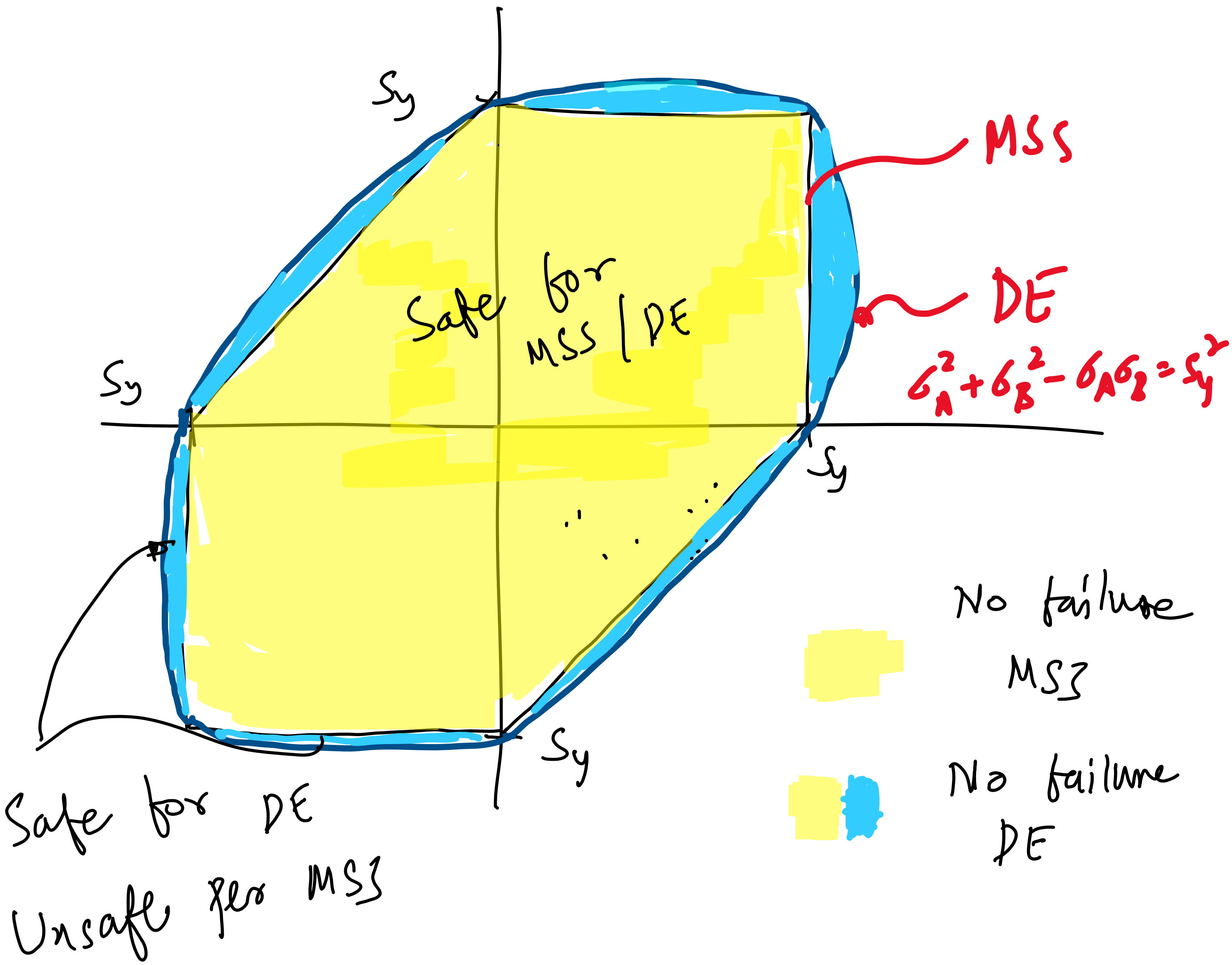
$$\sigma' = \left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2}$$

$$\sigma' = \left[\frac{(\sigma_A - \sigma_B)^2 + (\sigma_B - 0)^2 + (0 - \sigma_A)^2}{2}\right]^{1/2}$$

$$\boxed{\sigma' = \left(\sigma_A^2 + \sigma_B^2 - \sigma_A \sigma_B\right)^{1/2} \geq S_y}$$

2D failure criteria per DE

EQUATION of AN ELLIPSE



MSS is more conservative than DE

Design using DE

$$n_d \,|\, n = \frac{S_y}{\sigma'}$$

Von-Mises based on σ_x, σ_y, τ_{xy}

$$\sigma' = \sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2} \quad — (1)$$

$$\left.\begin{aligned} \sigma_{1,2} &= \frac{\sigma_x+\sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \sigma_3 &= 0 \end{aligned}\right\} \quad (2)$$

Substitute (2) in (1)

$$\sigma' = \left[\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2\right]^{1/2} \quad -(3)$$

Using PE for shear experiment

Put $\sigma_x = \sigma_y = 0$ in (3) $\Rightarrow \sigma' = \left(3\tau_{xy}^2\right)^{\frac{1}{2}}$

$$\sigma' = 1.732\ \tau_{xy}$$

Put $\sigma_x = S_y$; $\sigma_y = \tau_{xy} = 0$ in (3)

$$\sigma' = (S_y)$$

$$1.732\ \tau_{xy} = S_y$$

$$\boxed{\tau_{xy} = 0.5773\ S_y}$$

S_{sy} = yield strength in shear

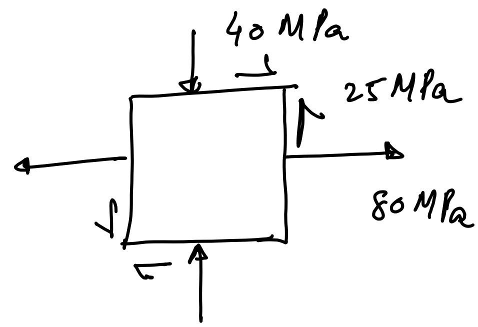
$$\boxed{S_{sy} = 0.5773\ S_y \Rightarrow \tau_{xy} = S_{sy}}$$

Example

Assume $S_y = 250$ MPa

Compute the factor of safety assumming

1. yield due to MSS
2. yield due to DE



MSS : $n = \frac{S_y}{(\sigma_1 - \sigma_3)}$

DE : $n = \frac{S_y}{\sigma'} = \frac{S_y}{(\sigma_A^2 + \sigma_B^2 - \sigma_A \sigma_B)^{1/2}}$

Compute principle stresses.

$$\sigma_{a,b} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma_{a,b} = 85, -45$; $\sigma_c = 0$

$\sigma_1 = 85$; $\sigma_3 = -45$

(a) MSS

$$n = \frac{S_y}{\sigma_1 - \sigma_3} = \frac{250}{85 - (-45)} = \frac{250}{130} = 1.92$$

* $n_{MSS} = 1.92$ more conservative

(b) DE

$$n = \frac{S_y}{(\sigma_A^2 + \sigma_B^2 - \sigma_A \sigma_B)^{1/2}} = \frac{250}{(85^2 + (-45)^2 - (85)(-45))^{1/2}}$$

$$n = \frac{250}{114.35} = 2.18$$

* $n_{DE} = 2.18$