Jacobian and its applications

Lets say we have a function

$$f = [f_1(q), f_2(q), f_3(q), \ldots f_m(q)]$$

Here $q = [x_1, x_2, x_3, \ldots x_n)$

$$J = \frac{\partial f}{\partial q} = \frac{\partial(f_1, f_2, \ldots f_m)}{\partial(x_1, x_2, \ldots x_n)} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}}_{m \times n}$$

Example:

$$f = [\underline{x^2 + y^2}, \underline{\underline{2x + 3y}} + 5] = [f_1, f_2)$$

$$q = [x, y]$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x, & 2y \\ 2, & 3 \end{bmatrix}$$

At $x = 1, y = 2$ $\quad J = \begin{bmatrix} 2, & 4 \\ 2 & 3 \end{bmatrix}$

In MATLAB

J_s = jacobian (f, q) [symbolic]

J_n = subs (J_s, [1 2]) [numeric]

or

J_n = finite difference

$$= \begin{bmatrix} \frac{\Delta f_1}{\Delta x} & \frac{\Delta f_1}{\Delta y} \\ \frac{\Delta f_2}{\Delta x} & \frac{\Delta f_2}{\Delta y} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{f_1(x+\Delta x, y) - f_1(x,y)}{\Delta x} & \frac{f_1(x, y+\Delta y) - f_1(x,y)}{\Delta y} \\ \vdots & \vdots \end{bmatrix}$$

see MATLAB.

Application of the jacobian: 1) Finding velocity

Derivation

$$J = \frac{\partial f}{\partial q} \qquad \text{definition}$$

$$r = f(q) \qquad \text{e.g. } x = l\cos\theta$$

$$\frac{\partial r}{\partial q} = \frac{\partial f}{\partial q} = J$$

$$\Rightarrow \partial r = J\,\partial q$$

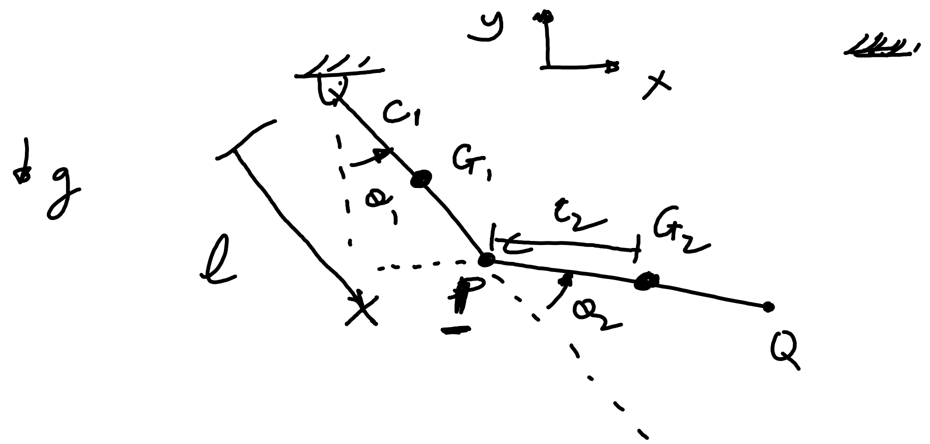
Divide by ∂t

$$\frac{\partial r}{\partial t} = J\frac{\partial q}{\partial t}$$

$$\frac{dr}{dt} = J\frac{dq}{dt}$$

$$\boxed{\dot{r} = J\,\dot{q}}$$

Example : Double pendulum



$$v_{G_1} = \frac{d r_{G_1}}{dt} = J_{G_1} \dot{q}$$

$$q = [\underline{\theta_1}, \underline{\theta_2}]$$

$$r_{G_1} = [c_1 \sin\theta_1, -c_1 \cos\theta_1] = [x_{G_1}, y_{G_1}]$$

$$J_{G_1} = \frac{\partial r_{G_1}}{\partial q} = \begin{bmatrix} \frac{\partial x_{G_1}}{\partial \theta_1} & \frac{\partial x_{G_1}}{\partial \theta_2} \\ \frac{\partial y_{G_1}}{\partial \theta_1} & \frac{\partial y_{G_1}}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} c_1 \cos\theta_1 & 0 \\ c_1 \sin\theta_1 & 0 \end{bmatrix}$$

$$v_{G_1} = \begin{bmatrix} c_1 \cos\theta_1 & 0 \\ c_1 \sin\theta_1 & 0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} c_1 \omega_1 \cos\theta_1 \\ c_1 \omega_1 \sin\theta_1 \end{bmatrix}$$

$$v_{G_2} = J_{G_2} \dot{q} = \begin{bmatrix} \frac{\partial x_{G_2}}{\partial \theta_1} & \frac{\partial x_{G_2}}{\partial \theta_2} \\ \frac{\partial y_{G_2}}{\partial \theta_1} & \frac{\partial y_{G_2}}{\partial \theta_2} \end{bmatrix}$$

$$x_{G_2} = l \sin\theta_1 + c_2 \sin(\theta_1 + \theta_2)$$

$$y_{G_2} = -l\cos\theta_1 - c_2\cos(\theta_1 + \theta_2)$$

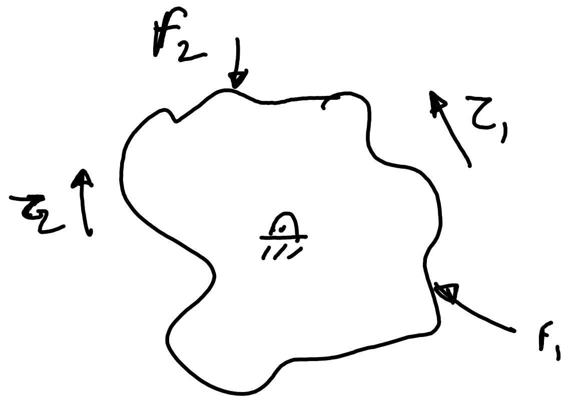
$$\vdots$$

$$v_{G_2} = \ \ldots$$

$$v_p = J_p\,\dot{q} = \begin{bmatrix} \frac{\partial x_p}{\partial \theta_1} & \frac{\partial x_p}{\partial \theta_2} \\ \frac{\partial y_p}{\partial \theta_1} & \frac{\partial y_p}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Application of Jacobian: 2) Static forces

Derivation



$$\underbrace{\delta W}_{\text{work done}} = \tau^T \delta\theta - F^T \delta r$$

$$\delta W = 0$$

$$\tau^T \delta\theta - F^T \delta r = 0$$

$$\tau^T \delta\theta = F^T \delta r$$

$$\Rightarrow \quad \tau^T = F^T \frac{\delta r}{\delta\theta} = F^T J$$

$$\Rightarrow \quad \tau^T = F^T J$$

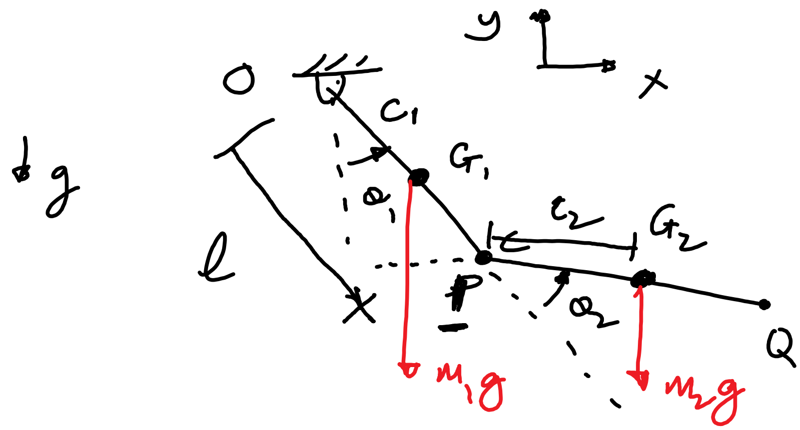
Take transpose on both sides

$$(\tau^T)^T = (F^T J)^T \qquad\qquad (AB)^T = B^T A^T$$

$$\tau = J^T F$$

$$\boxed{\tau = J^T F}$$

Example : Double pendulum



Find the torques τ_1 & τ_2 needed at O and P such that the pendulum is in static equilibrium

$$\tau = J^T \underline{\underline{F}}$$

$$\underline{\underline{\tau}} = J_{G_1}^T F_{G_1} + J_{G_2}^T F_{G_2}$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \underline{\underline{J_{G_1}^T}} \begin{bmatrix} 0 \\ \underline{\underline{-m_1 g}} \end{bmatrix} + \underline{\underline{J_{G_2}^T}} \begin{bmatrix} 0 \\ \underline{\underline{-m_2 g}} \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \cos\theta_1 & c_1 \sin\theta_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix} + \dots$$

$$\begin{bmatrix} c_2 \cos(\theta_1+\theta_2) + l\cos\theta_1 & c_2 \sin(\theta_1+\theta_2) + l\sin\theta_1 \\ c_2 \cos(\theta_1+\theta_2) & c_2 \sin(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} -m_1 g\, c_1 \sin\theta_1 - m_2 g\, c_2 \sin(\theta_1+\theta_2) - m_2 g\, l \sin\theta_1 \\ -m_2 g\, c_2 \sin(\theta_1+\theta_2) \end{bmatrix}}$$