## Dynamics

- Newton-Euler's method:
  - Free Body Diagram
  - $F = m\dot{v}$ $\quad \dot{H} = r \times m\dot{v} + I\alpha$

- Euler-Lagrange method
  - No Free Body diagram
  - We only need the kinematics.

- Kane's methods ....

## Euler-Lagrange Equations

1) Find the position of the center of mass of each link in the global frame. $[x_c^0, y_c^0]$

Find the velocity of the center of mass $[\dot{x}_c, \dot{y}_c)$ where $\cdot$ is $\frac{d}{dt}$

2) Find the Lagrangian $\mathcal{L} = T - V$
   $T \rightarrow$ kinetic energy $\quad V$ — potential energy

$\rightarrow T = \sum_{i=1}^{n} 0.5\, m v_i^2 + 0.5\, I_i \omega_i^2$

$\rightarrow V = \sum_{i=1}^{n} \left(m_i\, g_i\, (y_c^0)_i\right) + 0.5 \sum_{p=1}^{q} k\left(r_p - r_{p0}\right)^2$

$r_p$ is the stretched length of the spring
$r_{p0}$ is unstretched length of the spring

3) Write the equations of motion

$$\rightarrow \quad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$q \rightarrow$ degrees of freedom

projectile $q = \{x, y\}$

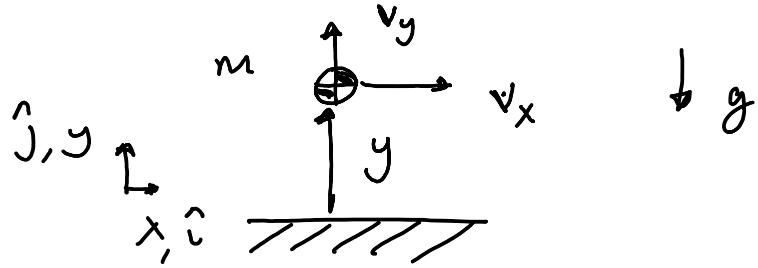
2-link pendulum $q = \{\theta_1, \theta_2\}$

1 projetile such as a rod $q = \{x, y, \theta\}$

$Q \rightarrow$ external forces.

4) Solve $\underline{\underline{\ddot{x}_c, \ddot{y}_c, \ddot{\theta}}}$

Example – projectile




The projectile is subject to a drag force

$$\vec{F} = - c v^2 \hat{v} = - c(\dot{x}^2+\dot{y}^2)\left(\frac{\dot{x}\hat{i} + \dot{y}\hat{j}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right)$$

$$\vec{F} = \left(-c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x}\,\hat{i}\right) + \left(-c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}\right)\hat{j}$$

$\Rightarrow$ Find the equations of motion & simulate the projectile

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1) position: $x, y$

velocity: $\dot{x}, \dot{y}$

2) $\mathcal{L} = T - V = \frac{1}{2} m v^2 - mgy$

$$\mathcal{L} = \frac{1}{2} m(\dot{x}^2+\dot{y}^2) - mgy$$

3) $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q} = Q_j$

$q = x, y$

$$\mathcal{L} = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = F_x$$

$$\frac{d}{dt}\left(\cancel{\tfrac{1}{2}} m(\cancel{2}\dot{x} + 0) - 0\right) - 0 = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x}$$

$$m\ddot{x} = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x}$$

$$\rightarrow \boxed{\ddot{x} = -\frac{c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x}} \leftarrow (1)$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = F_y$$

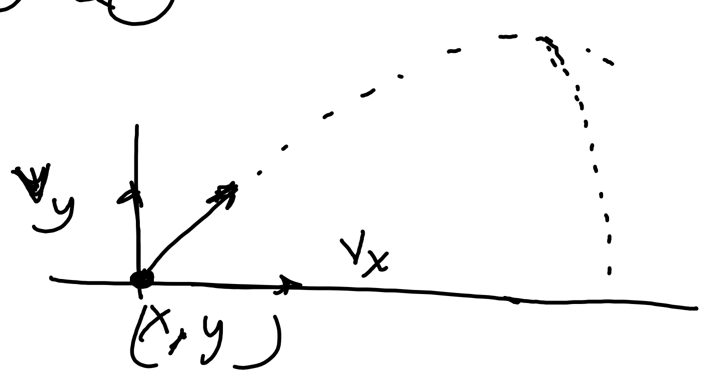
$$\frac{d}{dt}\left(\cancel{\tfrac{1}{2}} m(\cancel{2}\dot{y})\right) - (-mg) = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}$$

$$m\ddot{y} + mg = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}$$

$$\rightarrow \boxed{\ddot{y} = -g - \frac{c}{m}\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}} - (2)$$

4) $\ddot{x}, \ddot{y} \rightarrow (1)\ \&\ (2)$

Simulation



# Differentiation

$\frac{\partial L}{\partial q}$, $\frac{\partial L}{\partial \dot{q}}$, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$

MATLAB: diff ( $\underline{L}$ , q) = $\frac{\partial L}{\partial q}$

e.g. syms $\underline{x}$ $\underline{\text{real}}$

f = $\underline{x^2 + 2x + 1}$

dfdx = diff (f, x) : 2x+2

subs (dfdx, 1) : $\underline{\underline{4}}$

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$$\frac{df}{dx} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$\Delta x$ is small.

① $$\frac{df}{dx} = \frac{f(1+\Delta x) - f(1)}{\underline{\Delta x}} =$$ $\Delta x = 1e^{-3}$ (forward difference)

② $$\frac{df}{dx} = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}$$ $\Delta x = 1e^{-3}$ (central difference)

$x = 1$

# Chain rule

$f_1(x(t))$

$$f_1 \longrightarrow x \longrightarrow t$$

$$\frac{df_1}{dt} = \frac{df_1}{dx}\,\frac{dx}{dt} \qquad \text{chain rule}$$

e.g. $f_1(x(t)) = \sin(x(t))$

$$\frac{df_1}{dt} = \frac{df}{dx}\,\frac{dx}{dt} = \cos(x)\,\frac{dx}{dt}$$

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$f_2(x(t), \dot{x}(t))$

$$f_2 \longrightarrow x, \dot{x} \longrightarrow t$$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\,\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\,\frac{d\dot{x}}{dt}$$

$\frac{dx}{dt}$

e.g. $f_2(x, \dot{x}) = \underbrace{x(t)}\,\underbrace{\dot{x}(t)}$

$$\frac{df_2}{dt} = \frac{df_2}{dx}\,\frac{dx}{dt} + \frac{df_2}{d\dot{x}}\,\frac{d\dot{x}}{dt}$$

$$= \dot{x}(\dot{x}) + x\,\ddot{x}$$

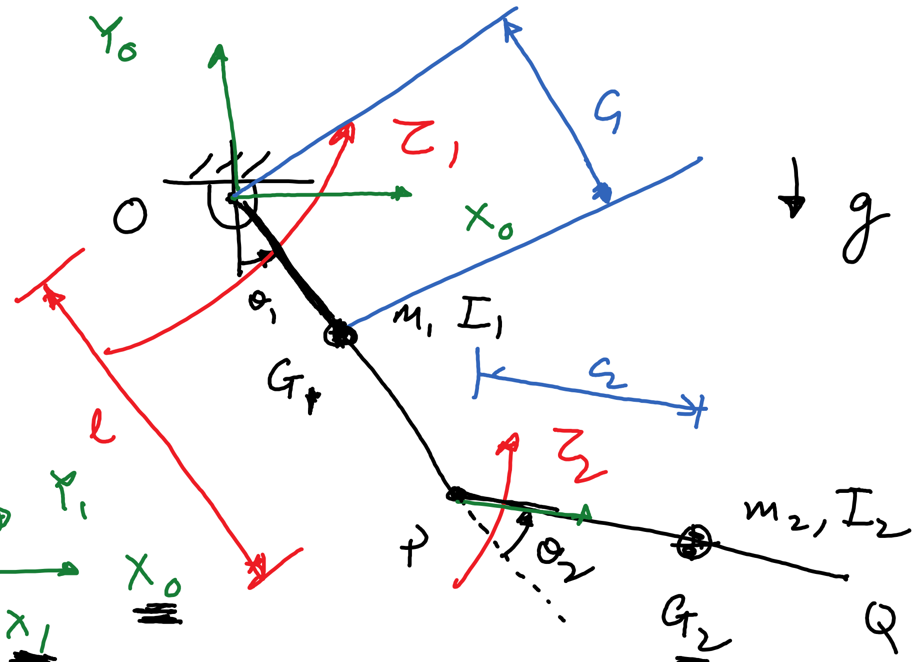
$$\frac{df_2}{dt} = \dot{x}^2 + x\ddot{x}$$

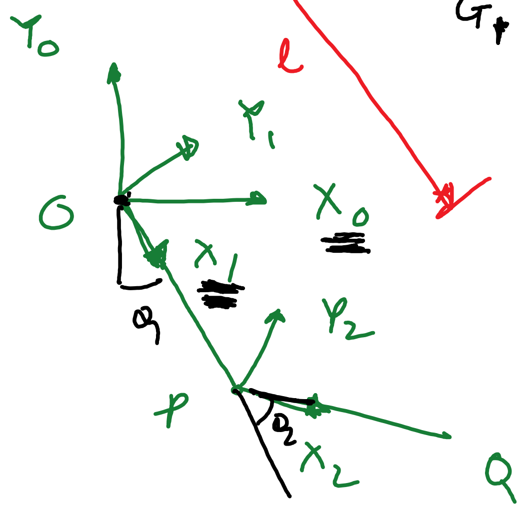
$f_1\ (x(t))$

$df_1dt\ =\ diff(f_1, x)\ \dot{x}$

$f_2\ (x, \dot{x})$

$df_2dt = diff(f_2, x)\,\dot{x} + diff(f_2, \dot{x})\,\ddot{x}$



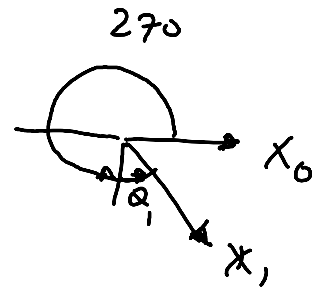




⟹ Find the equations of motion, simulate, animate the system

(1) $G_1^0 = H_1^0 \, \underline{G}_1^1$

$$H_1^0 = \begin{bmatrix} \underline{R}_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} \cos(270+\theta_1) & -\sin(270+\theta_1) \\ \sin(270+\theta_1) & \cos(270+\theta_1) \end{bmatrix}$$



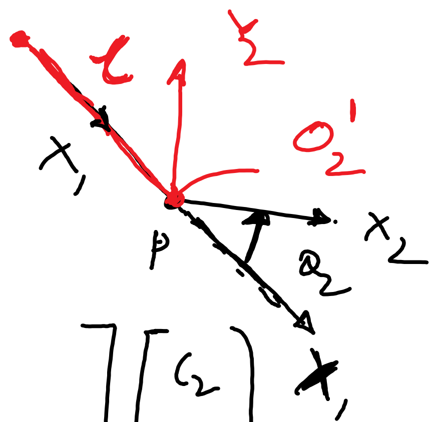

$$O_1^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_{G_1}^0 \\ y_{G_1}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta_1 & \cos\theta_1 & 0 \\ -\cos\theta_1 & \sin\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{c_1} \\ \underline{0} \\ 1 \end{bmatrix}$$

$$G_2^0 = H_1^0 \, H_2^1 \, G_2^2$$

$$H_2^1 = \begin{bmatrix} R_2^1 & O_2^1 \\ 0 & 1 \end{bmatrix} \qquad G_2^2 = \begin{bmatrix} c_2 \\ 0 \\ 1 \end{bmatrix}$$

$$H_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & l \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



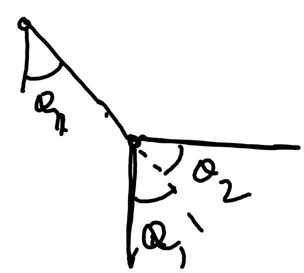

$$\begin{bmatrix} x^0_{G_2} \\ \underline{\underline{y}}^0_{G_2} \\ 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} c_2 \\ 0 \\ 1 \end{bmatrix}$$

$$V^0_{G_1} = \begin{bmatrix} \dot{x}^0_{G_1} \\ \dot{y}^0_{G_1} \end{bmatrix}_{2\times 1} \qquad V^0_{G_2} = \begin{bmatrix} \dot{x}^0_{G_2} \\ \dot{y}^0_{G_2} \end{bmatrix}$$

② $$\underline{\underline{T}} = 0.5\, m_1 \Big( \underbrace{\underline{V}^{0\,T}_{G_1}}_{1\times 2} \; \underbrace{\underline{V}^0_{G_1}}_{2\times 1} \Big) + 0.5\, m_2 \, \underline{\underline{V}}^{0\,T}_{G_2} \, \underline{V}^0_{G_2}$$

$$+ 0.5\, I_1 \, \underline{\underline{\dot{\theta}_1^2}} + 0.5\, I_2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$\underline{V} = m_1 g \, \underline{\underline{y}}^0_{G_1} + m_2 g \, \underline{\underline{y}}^0_{G_2}$$



$\mathcal{L} = T - V$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j \quad \Big\} \text{ MATLAB}$$

$q_j = \theta_1, \theta_2$

$Q_j = \tau_1, \tau_2$

(4) EOM1 — $q_j = \theta_1$ $\quad\quad \ddot{x} =$ ______ ✓

EOM2 — $q_j = \theta_2$ $\quad\quad \ddot{y} =$ ______ ✓

Manipulator

$$M(\theta)\ddot{\theta} + C(\underline{\theta}, \dot{\theta})\dot{\theta} + \underline{G}(\theta) = \tau$$

$M(\theta)$ — mass matrix

$\theta$ — $[\theta_1 \quad \theta_2]^T$

$C(\theta, \dot{\theta})\dot{\theta}$ — coriolis & centripetal acceleration

$\underline{G}(\theta)$ — gravity

$$\ddot{\theta} = \underline{\underline{M}}^{-1}(\tau - C - G)$$ ✓

$\boxed{\text{DONT DO THIS SYMBOLICALLY}}$

$M, \tau, C, G$

✓ ✓ ✓ ✓

$M\ddot{\theta} = \tau - C - G$

numeric

numeric

Inversion is done numerically $\quad \ddot{\theta} = M^{-1}(\tau - C - G)$

$\underline{\underline{M}}$ $\quad$ inv(M)

EOM1, EOM2

$$EOM = M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) - \tau$$

$$\text{subs}(EOM, [\ddot{\theta}], [0]) \rightarrow \underbrace{C(\theta,\dot{\theta})\dot{\theta} + G(\theta) - \tau}_{= B}$$

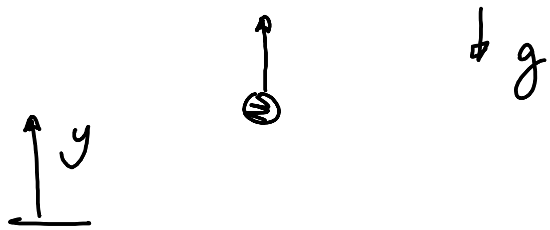
$$\text{jacobian}(EOM, \ddot{\theta}) \rightarrow M$$

next class

$$\text{subs}(B, [\dot{\theta}], [0]) \rightarrow G(\theta)$$

$$\text{subs}(B, \dot{\theta}, 0) - G \rightarrow C$$

Euler-Lagrange




$$\mathcal{L} = T - V = \frac{1}{2} m\dot{y}^2 - mgy$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathcal{L}}{\partial q_j} = Q_j$$

$$q_j = y \qquad Q_j = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = -mg \; ; \qquad \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{1}{2} m (2\dot{y}) - 0 = m\dot{y}$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) = \frac{d}{dt}\left(m\dot{y}\right) = m\ddot{y}$$

$$m\ddot{y} - (-mg) = 0$$

$$\boxed{\ddot{y} = -g}$$