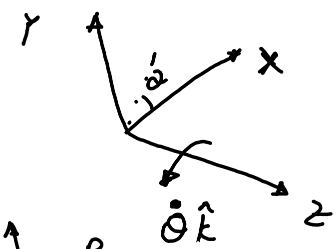
# 3D angular velocity

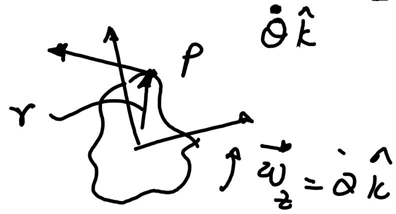
In 2D




Fact 1 $\vec{\omega}_z = \dot{\theta}\hat{k}$

angular speed

Fact 2 $\vec{v}_p = \vec{\omega}_z \times \vec{r}$




Linear speed

cross product

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In 3D $\phi \to x;\ \theta \to y;\ \psi \to z$

$\vec{\omega} \neq \dot{\phi}\hat{i} + \dot{\theta}\hat{j} + \dot{\psi}\hat{k}$ Fact 1 does not hold true in 3D

euler rates.

$\vec{v} = \vec{\omega} \times \vec{r}$ Fact 2 hold true in 3D

How to get $\vec{\omega}$ in 3D?

Skew symmetric matrices (S)

$$S(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

$$a = [a_1 \quad a_2 \quad a_3]$$

<u>Properties :</u>

1) $S(a) + S^T(a) = 0$

2) $\underbrace{\underset{3\times1}{\vec{a}} \times \underset{3\times1}{\vec{b}}}_{3\times1} = \underset{\substack{\text{matrix} \\ 3\times3}}{S(a)} \, \underset{\substack{\text{vector} \\ 3\times1}}{b}$

3X1

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

3) $R\,S(a)\,R^T = S(R\,a)$ only if $R$ is a rotation matrix

⇒ Proof is in Spong's book

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \hat{i}\,(a_2 b_3 - a_3 b_2) - \hat{j}\,(a_1 b_3 - b_1 a_3) + \hat{k}\,(a_1 b_2 - a_2 b_1)$$

$$\vec{a} \times \vec{b} = \begin{bmatrix} a_2 b_3 - a_3 b_2 \\ b_1 a_3 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{bmatrix} \begin{matrix} \checkmark \\ \checkmark\checkmark \\ \checkmark\checkmark\checkmark \end{matrix}$$

$$S(a)\,b = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -a_3 b_2 + a_2 b_3 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix} \begin{matrix} \checkmark \\ \checkmark\checkmark \\ \checkmark\checkmark\checkmark \end{matrix}$$

$$\vec{a} \times \vec{b} = S(a)\,b \quad \checkmark$$

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$$C^0 = \underbrace{R_z R_y R_x}\, C^3$$

$$\rightarrow \quad r = \underline{R}\; \underline{r_b}$$

$r_b$ – body frame
$r$ – world frame
$R = R_z(\psi) R_y(\theta) R_x(\phi)$

Diff. w.r.t. time $(t)$

$$\underline{\underline{\dot{r} = \dot{R}\, r_b + R\, \cancel{\dot{r}_b}^{\,0}}} \quad \text{—} \; \mathrm{I}$$

$$R R^T = I$$

Diff. wrt time

$$\dot{R} R^T + R \dot{R}^T = 0$$

$$\dot{R} R^T + \left[ \left( R \dot{R}^T \right)^T \right]^T = 0$$

$$\dot{R} R^T + \left( \left( \dot{R}^T \right)^T R^T \right) = 0 \qquad (AB)^T = B^T A^T$$

$$\dot{R} R^T + \left( \dot{R} R^T \right)^T = 0$$

$S(a)$ – skew symmetric matrix

$$S(a) + S^T(a) = 0$$

$$\boxed{S(a) = \dot{R} R^T}$$

$R$ is the rotation matrix

$$S(a) = \dot{R} R^T$$

Post multiply by $R$

$$S(a) R = \dot{R} \underbrace{\left( R^T R \right)}_{= I}$$

$$\boxed{\dot{R} = S(a) R} \quad \text{— (II)}$$

$$\dot{r} = \dot{R}\, r_b \qquad \{\, r = R\, r_b \,\} \text{ --- (III)}$$

$$\dot{R} = S(a) R \quad \text{- from (II)}$$

$$\dot{r} = S(a)\, \underbrace{R\, r_b}$$

$$\dot{r} = S(a)\, \underset{=b}{r} \quad \text{from (III)}$$

(IV) $$\dot{r} = \vec{a} \times \vec{r} \implies \vec{a} \times \vec{b} = S(a)\, b$$

But we know that $\vec{v} = \vec{\omega} \times \vec{r}$

(V) $$\dot{r} = \vec{\omega} \times \vec{r}$$

From (IV) and (V) we can see that

$$a = \omega$$

$$\boxed{\dot{R} R^T = S(\omega) \qquad \dot{R} = S(\omega) R}$$

$$S(\omega) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

This shows how $\omega$ is related to $R = R_z R_y R_x$
$= R_z(\psi) R_y(\theta) R_x(\phi)$

But how is $\omega$ related to $\underline{\underline{\dot{\psi}, \dot{\theta}, \dot{\phi}}}$ ?

$$\dot{R} = S(\omega) R$$

$$\dot{R} R^T = S(\omega) \underbrace{R R^T}_{= I}$$

$$\Rightarrow \quad \underset{?}{S(\omega)} = \dot{R} R^T \qquad \Rightarrow R = R_z(\psi) R_y(\theta) R_x(\phi)$$

$$= \frac{d}{dt}(R_z R_y R_x) R^T$$

$$S(\omega) = \underbrace{\dot{R}_z R_y R_x R^T}_{1} + \underbrace{R_z \dot{R}_y R_x R^T}_{2} + \underbrace{R_z R_y \dot{R}_x R^T}_{3}$$

1 $\quad \dot{R}_z R_y R_x R^T = \dot{R}_z R_y R_x (R_z R_y R_x)^T$

$$= \dot{R}_z \underbrace{R_y \underbrace{R_x R_x^T}_{=I} R_y^T}_{=I} R_z^T$$

$$= \dot{R}_z R_z^T$$

$$=$$

$$\dot{R} = S(\omega) R$$

$$\dot{R}_z = S(\omega_z) R_z$$

$$\dot{R}_z = S(\dot{\psi} k) R_z$$

$$\dot{R}_z R_z^T = S(\dot{\psi} k)$$

$$= S(\dot{\psi} k)$$

(2) $R_z \dot{R}_y R_x R^T = R_z \dot{R}_y R_x (R_z R_y R_x)^T$

$$= R_z \dot{R}_y \underbrace{R_x R_x^T}_{= I} R_y^T R_z^T$$

$$= R_z \dot{R}_y R_y^T R_z^T$$

$\dot{R} = S(\omega) R$

$\dot{R}_y = S(\omega_y) R_y$

$\dot{R}_y = S(\dot{\theta} \hat{j}) R_y$

$\dot{R}_y R_y^T = S(\dot{\theta} \hat{j})$

$$= R_z S(\dot{\theta} \hat{j}) R_z^T$$

Property

$R S(a) R^T = S(Ra)$

$$= S(R_z \dot{\theta} \hat{j})$$

(3) $R_z R_y \dot{R}_x R^T = R_z R_y \dot{R}_x (R_z R_y R_x)^T$

$$= R_z R_y \dot{R}_x R_x^T R_y^T R_z^T$$

$\dot{R} = S(\omega) R$

$\dot{R}_x = S(\omega_x) R_x$

$\dot{R}_x = S(\dot{\phi} \hat{i}) R_x$

$\dot{R}_x R_x^T = S(\dot{\phi} \hat{i})$

$$= R_z R_y S(\dot{\phi} \hat{i}) \underbrace{R_y^T R_z^T}$$

$$= \underbrace{(R_z R_y)} S(\dot{\phi} \hat{i}) \underbrace{(R_z R_y)}^T$$

Property

$R S(a) R^T = S(Ra)$

$$= S(R_z R_y \dot{\phi} \hat{i})$$

$$S(\omega) = S(\dot{\psi}\hat{k}) + S(R_z\dot{\theta}\hat{j}) + S(R_z R_y \dot{\phi}\hat{i})$$
$$S(\omega) = S(\dot{\psi}\hat{k} + R_z\dot{\theta}\hat{j} + R_z R_y \dot{\phi}\hat{i})$$

$$\boxed{\omega = \dot{\psi}\hat{k} + R_z\dot{\theta}\hat{j} + R_z R_y \dot{\phi}\hat{i}}$$

angular velocity in fixed frame

$$\omega = \dot{\psi}\begin{bmatrix}0\\0\\1\end{bmatrix} + R_z\dot{\theta}\begin{bmatrix}0\\1\\0\end{bmatrix} + R_z R_y \dot{\phi}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$\omega = \begin{bmatrix}\cos\psi\cos\theta & -\sin\psi & 0\\ \cos\theta\sin\psi & \cos\psi & 0\\ -\sin\theta & 0 & 1\end{bmatrix}\begin{bmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix}$$

angular velocity in fixed frame

euler rates

$$\underline{\underline{\omega}} = A\,\dot{\Theta} \quad \Rightarrow \quad \dot{\Theta} = A^{-1}\omega$$

determinat of A

$$= \frac{\text{cofactors } A}{\det A}$$

$$= \underline{\underline{\cos\theta}}$$

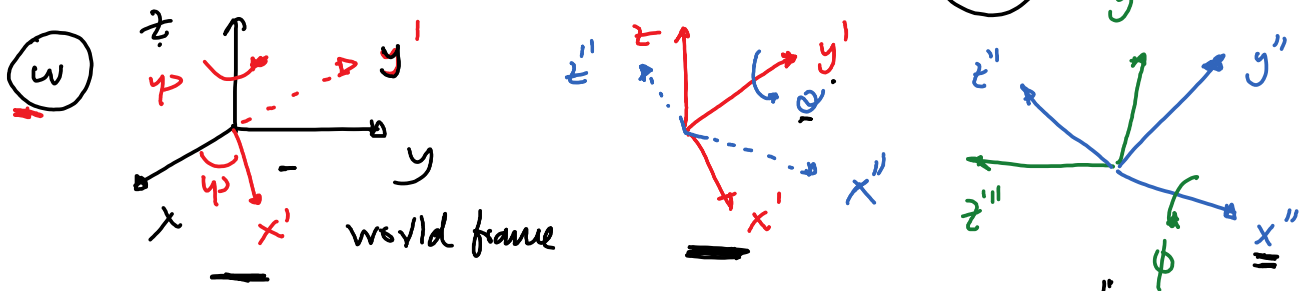
When $\theta = \pi/2$ then $A^{-1}$ is not defined

This is called as gimbal lock.

Problem with using euler angles / sol$^n$ use quaternions.

Simpler way of deriving $\omega$. $\quad \Omega$ — body frame

$\omega$




$$\omega = \dot{\psi}\, z + \dot{\theta}\, y' + \dot{\phi}\, \underline{\underline{x''}} = R_z R_y \hat{i}$$

$$= \dot{\psi}\,\hat{k} + \dot{\theta}\, R_z \hat{j} + \dot{\phi}\, R_z R_y \hat{i}$$

$$= \dot{\psi}\,\hat{k} + R_z \dot{\theta}\,\hat{j} + R_z R_y \dot{\phi}\,\hat{i}$$ — same formula as derived earlier

0 ——— 1 ——— 2 ——— 3

fixed frame $\quad R_z \quad R_y \quad R_x$

0 — 1 — 2 — 3 — 4 — . . . — n

$R_1^0 \quad R_2^1 \quad R_3^2 \quad R_4^3 \quad . . . \quad R_n^{n-1}$

$$\boxed{\omega_n^0 = \omega_1^0 + R_1^0\, \omega_2^1 + R_2^0\, \omega_3^2 + \dots R_{n-1}^0\, \omega_n^{n-1}}$$

Recursive formulae

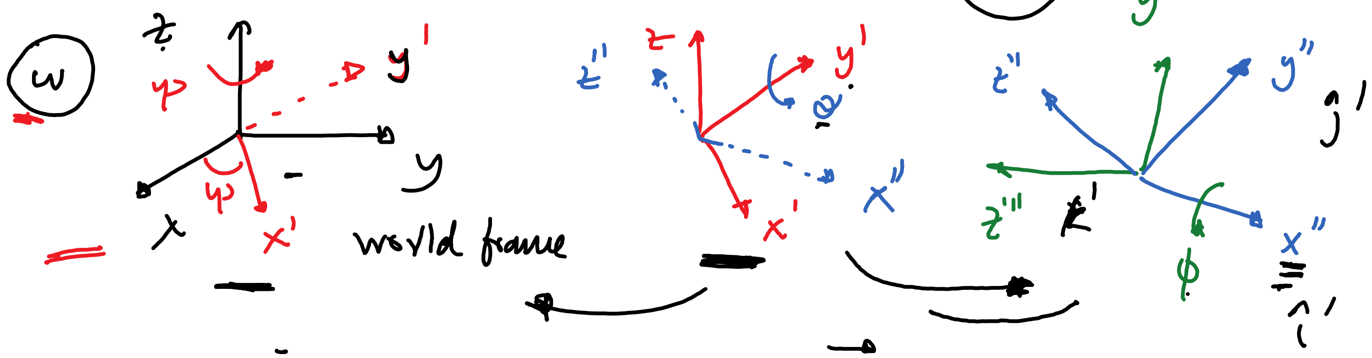
$$\boxed{\begin{aligned} \omega_2^0 &= \omega_1^0 + R_1^0\, \omega_2^1 \\ \omega_3^0 &= \omega_2^0 + R_2^0\, \omega_3^2 \\ \omega_4^0 &= \omega_3^0 + R_3^0\, \omega_4^3 \\ \omega_n^0 &= \omega_{n-1}^0 + R_{n-1}^0\, \omega_n^{n-1} \end{aligned}}$$

Useful for derivation.

## $\Omega$

## Angular velocity in body frame

$\Omega$ body frame




$$\Omega = \dot{\phi}\, x'' + \dot{\theta}\, y' + \dot{\psi}\, z$$

$$= \dot{\phi}\, x'' + \dot{\theta}\, R_x^T y'' + R_x^T R_y^T z''$$

$$\boxed{\Omega = \dot{\phi}\, \hat{i}' + \dot{\theta}\, R_x^T \hat{j}' + R_x^T R_y^T k'}$$

$\hat{i}', \hat{j}', k'$ are unit vectors in body frame

0 — 1 — 2 — 3 — . . . $n$

$$\boxed{\Omega_n = \omega_n^{n-1} + (R_n^{n-1})^T \omega_{n-1}^{n-2} + (R_{n-1}^{n-2})^T \omega_{n-2}^{n-3} \cdots (R_n^1)^T \omega_1^0}$$

or

$$\Omega_1 = \omega_1^0$$
$$\Omega_2 = \omega_2^1 + (R_2^1)^T \Omega_1$$
$$\Omega_3 = \omega_3^2 + (R_3^2)^T \Omega_2$$
$$\vdots$$
$$\Omega_n = \omega_n^{n-1} + (R_n^{n-1})^T \Omega_{n-1}$$

Recursive

$$\Omega = \dot{\phi}\,\hat{i}' + \dot{\theta}\, R_x^T\,\hat{j}' + R_x^T R_y^T\, k'$$

Expanding

$$\Omega = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$\det(\Omega) = \cos\theta$$ – true for 3-2-1

z-y-x