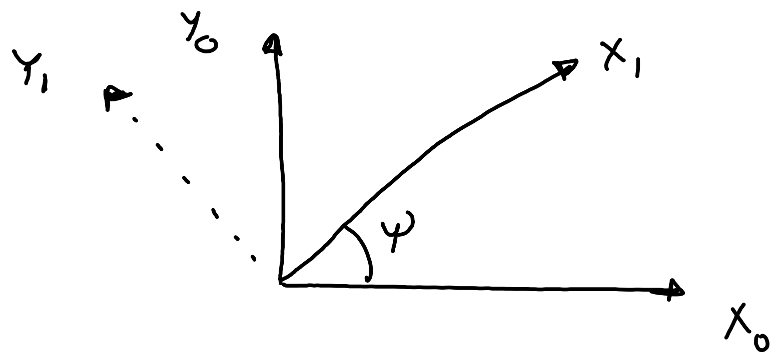
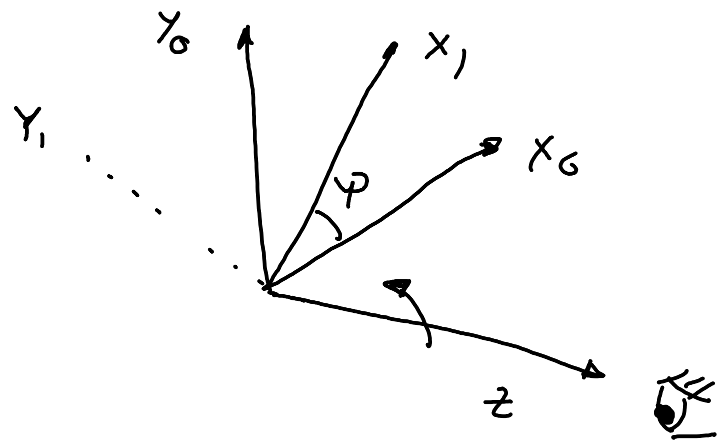
3D rotations and velocity





$$R_1^0 = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix}$$

2D rotation

$$R_z(\psi) = R_1^0 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3D rotation

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Rotation about all axis

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

9 numbers

Property: $R^T R = R R^T = I$ (I = identity) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

3
$$\left\{ \begin{array}{l} \sum_{i=1,2,3} r_{i1}^2 = 1 \Rightarrow r_{11}^2 + r_{21}^2 + r_{31}^2 = 1 \\ \sum_{i=1,2,3} r_{i2}^2 = 1 \\ \sum_{i=1,2,3} r_{i3}^2 = 1 \end{array} \right.$$

3
$$\left\{ \begin{array}{l} \sum_{i=1,2,3} r_{i1} r_{i2} = 0 \Rightarrow r_{11} r_{12} + r_{21} r_{22} + r_{31} r_{32} = 0 \\ \sum_{i=1,2,3} r_{i2} r_{i3} = 0 \\ \sum_{i=1,2,3} r_{i3} r_{i1} = 0 \end{array} \right.$$

9 numbers
− 6 constraints

= 3 unique number

We only need 3 numbers to describe 3D rotations.

Some of the ways to represent rotations

- Bryant angle

	$\boxed{x-y-z}$	$y-x-z$	$z-x-y$	
1-2-3	$x-z-y$	$y-z-x$	$\boxed{z-y-x}$	3-2-1
	$x-y-x$	$y-x-y$	$z-y-z$	Tait-Bryant angles?
	$x-z-x$	$y-z-y$	$z-x-z$	

12 combination

Euler angles.

Rotations in 2D are commutative

$$R_{\psi_1} = \begin{bmatrix} \cos\psi_1 & -\sin\psi_1 \\ \sin\psi_1 & \cos\psi_1 \end{bmatrix} \qquad R_{\psi_2} = \begin{bmatrix} \cos\psi_2 & -\sin\psi_2 \\ \sin\psi_2 & \cos\psi_2 \end{bmatrix}$$

ψ_1 followed by ψ_2

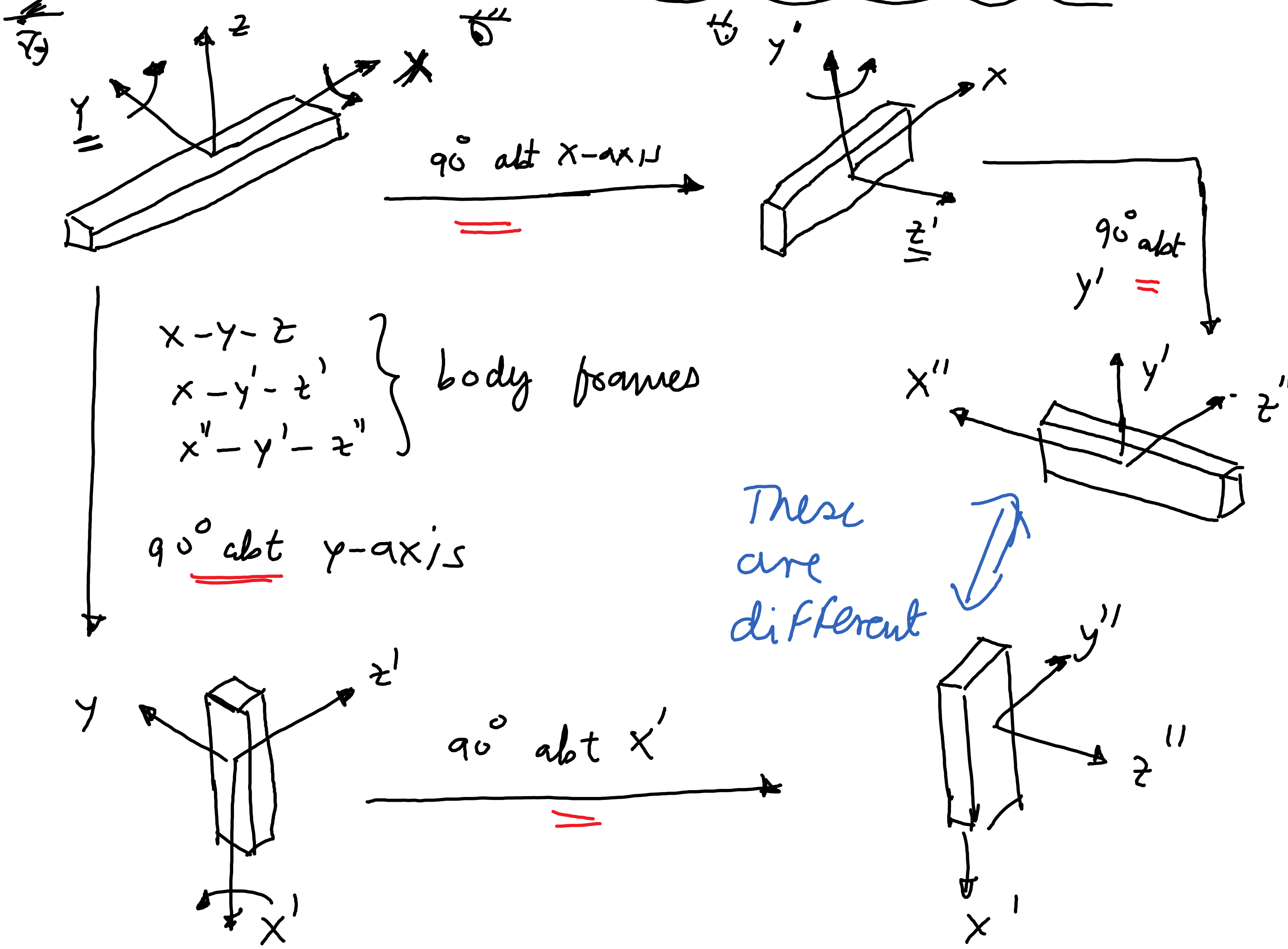
$$R_{\psi_1+\psi_2} = \begin{bmatrix} \cos(\psi_1+\psi_2) & -\sin(\psi_1+\psi_2) \\ \sin(\psi_1+\psi_2) & \cos(\psi_1+\psi_2) \end{bmatrix}$$

$$R_{\psi_1} R_{\psi_2} = R_{\psi_1+\psi_2} = R_{\psi_2} \cdot R_{\psi_1}$$

commutative

$ab = ba$

Rotations in 3D are NOT commutative

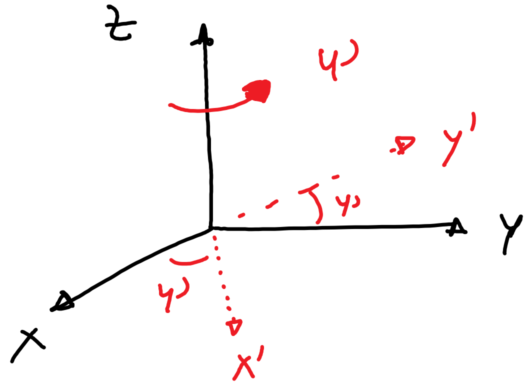


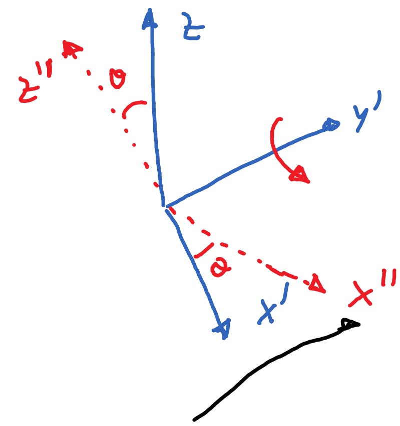
$$R_x R_y \neq R_y R_x$$

3-2-1 Euler angles $(z - y - x)$

$\psi - \theta - \phi$

psi theta phi





$c^0 \longrightarrow c^1$

$$\boxed{\underline{\underline{c}}^0 = R_z(\psi)\, \underline{\underline{c}}^1} \quad \text{I}$$

world frame Body frame

$c^1 \longrightarrow c^2$

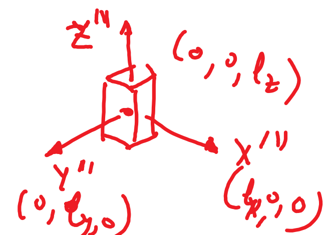
$$\boxed{c^1 = R_y(\theta)\, c^2} \quad \text{II}$$

intermediate frame G Body frame

Do this again abt. x''.

$c^2 \longrightarrow c^3$

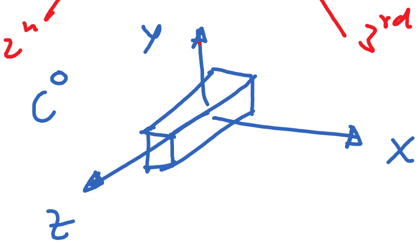
$$\boxed{c^2 = R_x(\phi)\, c^3} \quad \text{III}$$

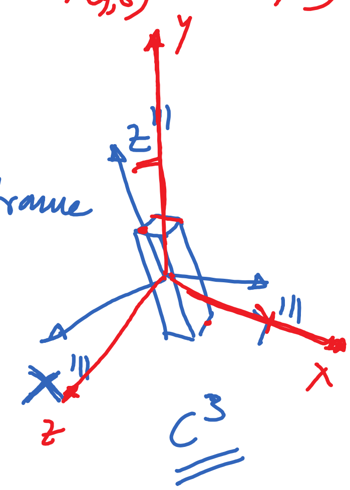


From I, II & III

$$\boxed{c^0 = R_z(\psi)\, R_y(\theta)\, R_x(\phi)\, c^3}$$

world frame 1st 2nd 3rd body frame





$$R = R_z(\psi)\, R_y(\theta)\, R_x(\phi)$$

$$= \begin{bmatrix} \cos\psi \cos\theta & \cos\psi \sin\phi \sin\theta - \cos\phi \sin\psi & \sin\psi \sin\phi + \cos\phi \cos\psi \sin\theta \\ \cos\theta \sin\psi & \cos\psi \cos\phi + \sin\phi \sin\psi \sin\theta & \cos\phi \sin\psi \sin\theta - \cos\psi \sin\phi \\ -\sin\theta & \cos\theta \sin\phi & \cos\phi \cos\theta \end{bmatrix}$$