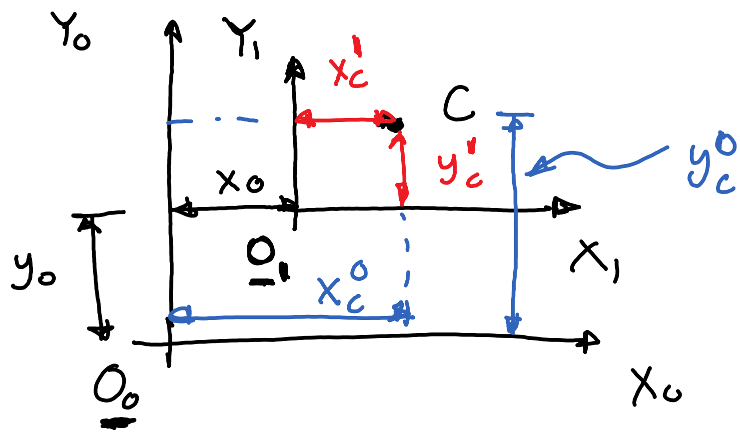
Coordinate frames: rotation and translation

1 Translation



$O_0 - X_0 - Y_0 \Rightarrow$ world frame / fixed frame

$O_1 - X_1 - Y_1 \Rightarrow$ local frame

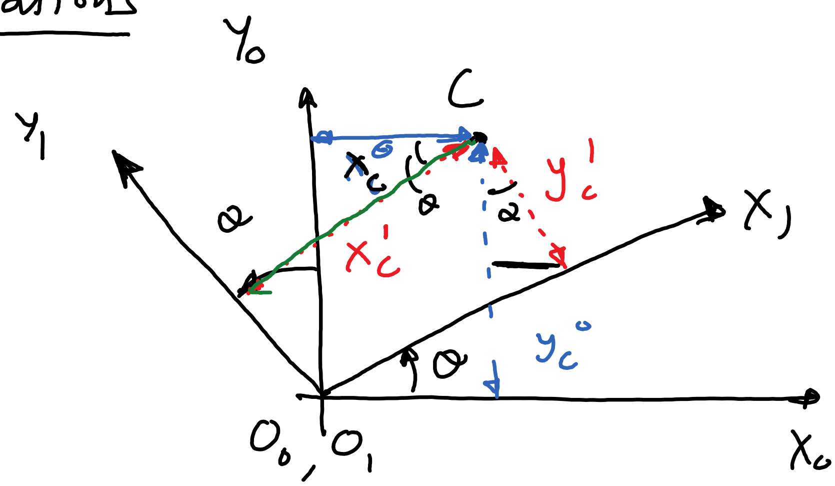
$\Rightarrow$ $C^1 = (x_c^1, y_c^1)$ [local]

$C^0 = (x_c^0, y_c^0)$ [world frame]

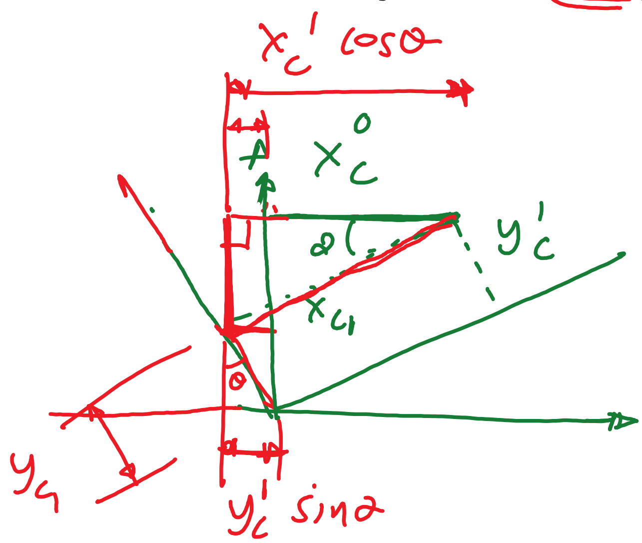
$O_1^0 = (x_0, y_0)$

$O_0^1 = (-x_0, -y_0)$

Rotations



$$x_c^0 = \cos\theta \; x_c^1 - \sin\theta \; y_c^1 \qquad - (1)$$

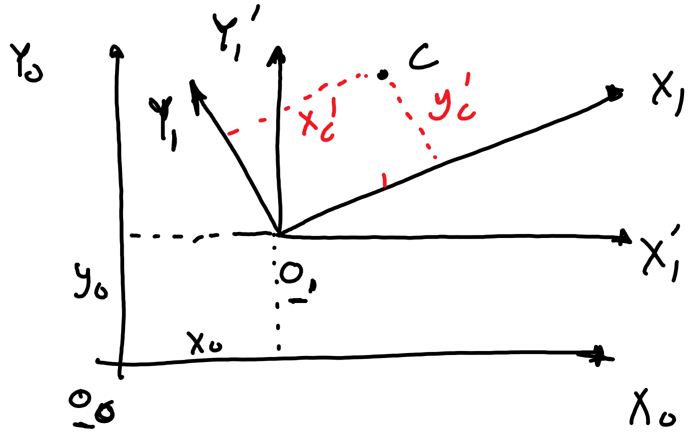


$$y_c^0 = \sin\theta \; x_c^1 + \cos\theta \; y_c^1 \qquad - (2)$$

$$\boxed{\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R_1^0} \begin{bmatrix} x_c^1 \\ y_c^1 \end{bmatrix} \quad c^0 = R_1^0 \, c^1}$$

$$\boxed{c^1 = (R_1^0)^{-1} c^0 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} c^0}$$

Combined rotation and translation



$$\begin{bmatrix} x_c^0 \\ y_c^0 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_c' \\ y_c' \end{bmatrix}$$

$$\boxed{c^0 = O_1^0 + R_1^0 \, c^1}$$

Homogenous transformations to represent combined rotation and translation

$$C^0 = O_1^0 + R_1^0 C^1 \qquad O_0X_0Y_0 \rightarrow O_1X_1Y_1$$

$$C^1 = O_2^1 + R_2^1 C^2 \qquad O_1X_1Y_1 \rightarrow O_2X_2Y_2$$

$$C^0 = O_1^0 + R_1^0 (O_2^1 + R_2^1 C^2)$$

$$\rightarrow \quad C^0 = O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 C^2$$

n-frames

$$C^0 = \underbrace{\left(O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 O_3^2 + \ldots R_1^0 R_2^1 R_3^2 \ldots R_{n-1}^{n-2} O_n^{n-1}\right)}_{\text{translation}} + \underbrace{\left(R_1^0 R_2^1 R_3^2 \ldots R_n^{n-1} C^n\right)}_{\text{rotation}}$$

Becomes complicates $\longrightarrow$ Homogenous transformation

solution

Homogenous transformation (H)

$$H_i^{i-1} = \begin{bmatrix} R_i^{i-1}{}_{2\times 2} & O_i^{i-1}{}_{2\times 1} \\ 0_{1\times 2} & 1_{1\times 1} \end{bmatrix}_{\underline{\underline{3\times 3}}}$$

$[0 \quad 0]$ (1)

$$C^i = \begin{bmatrix} c^i{}_{2\times 1} \\ 1_{1\times 1} \end{bmatrix}_{\underline{3\times 1}}$$

$$C^{i-1} = H_i^{i-1} C^i \qquad \left\{ \text{remember } c^{i-1} = R_i^{i-1} c^i \right\}$$

$$\begin{bmatrix} c^{i-1} \\ 1 \end{bmatrix} = \begin{bmatrix} R_i^{i-1} & O_i^{i-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c^i \\ 1 \end{bmatrix}$$

$$\boxed{C^0 = H_1^0 H_2^1 H_3^2 \ldots H_n^{n-1} C^n}$$

Let's see if this works out for $n=2$

$$C^0 = O_1^0 + R_1^0 O_2^1 + R_1^0 R_2^1 C^2$$

$$C^0 = H_1^0 H_2^1 C^2$$

$$\begin{bmatrix} C^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & O_2^1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C^2 \\ 1 \end{bmatrix}$$

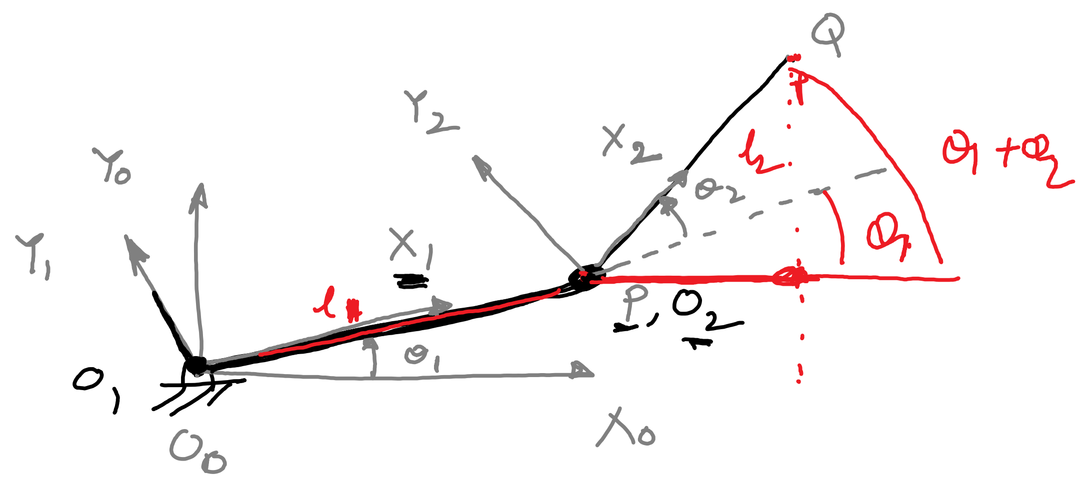
$$= \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 C^2 + O_2^1 (1) \\ 0\, C^2 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 C^2 + O_2^1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} C^0 \\ 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 C^2 + R_1^0 O_2^1 + O_1^0 \\ 0 + 1 \end{bmatrix}$$

Manipulator | Forward | kinematics

Given θ_1, θ_2 → find $\underline{p}^0, \underline{q}^0$



$$p^0 = H_1^0 \, p^1 \quad ; \quad Q^0 = H_1^0 H_2^1 Q^2$$

$$p^0 = \begin{bmatrix} R_1^0 & o_1^0 \\ 0 & 1 \end{bmatrix} p^1 \quad ; \quad Q^0 = \begin{bmatrix} R_1^0 & O_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & O_2^1 \\ 0 & 1 \end{bmatrix} Q^2$$

$$R_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \qquad R_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

$$O_1^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad O_2^1 = \begin{bmatrix} l_1 \\ 0 \end{bmatrix}$$

$$\mathbf{p}^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix} = \left\{ \begin{bmatrix} l_1 \\ 0 \\ 1 \end{bmatrix} \right. \qquad Q^2 = \begin{bmatrix} q^2 \\ 1 \end{bmatrix} = \begin{bmatrix} l_2 \\ 0 \\ 1 \end{bmatrix}$$

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_1 \cos\theta_1 \\ l_1 \sin\theta_1 \\ 1 \end{bmatrix}$$

$$Q^0 = \begin{bmatrix} q^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & l_1 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \begin{pmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{pmatrix} \\ 1 \end{bmatrix}$$