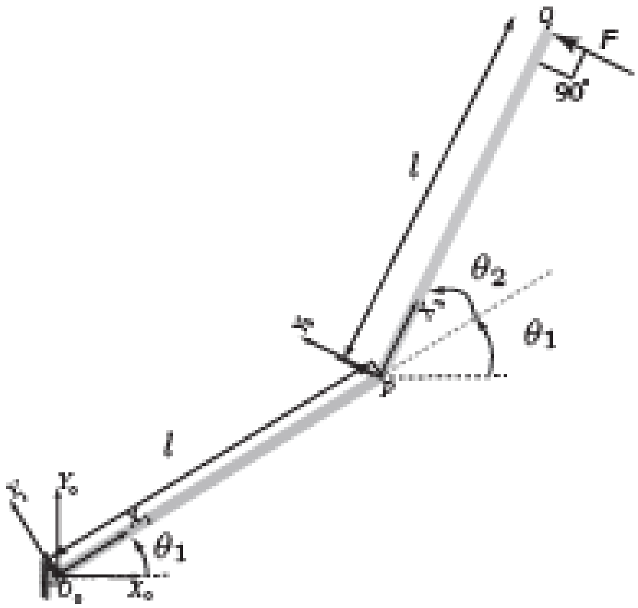


1. **Jacobian to find velocity:** Fig. 1 shows a two-link manipulator. Both link lengths are ℓ and the angles are θ_1 and θ_2 as shown. Let the velocity of point Q be $\mathbf{v}_Q$ and angular velocity of the two links is $\omega = [\dot{\theta}_1, \ \ \dot{\theta}_2] = [\omega_1, \ \ \omega_2]$.

   (a) Find the Jacobian of point Q, $\mathbf{J}_Q$. Write an analytical formula.

   (b) The velocity of point Q is given as $\mathbf{v}_Q = \mathbf{J}_Q\omega$. Thus, the angular velocity is given by $\omega = \mathbf{J}_Q^{-1}\mathbf{v}_Q$. Assume $\ell = 1$ and $\mathbf{v}_Q = [2; \ \ 1]$.

      i. Given $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{\pi}{3}$, find the numeric value of ω.

      ii. Given $\theta_1 = \frac{\pi}{6}$, $\theta_2 = 0$, find the numeric value of ω. Comment on the answer. Feel free to search the internet if you have to.

a) $$J_Q = \begin{bmatrix} \frac{\partial x_q}{\partial \theta_1} & \frac{\partial x_q}{\partial \theta_2} \\ \frac{\partial y_q}{\partial \theta_1} & \frac{\partial y_q}{\partial \theta_2} \end{bmatrix}$$ (Intuition?)

b) $V_Q = J_Q \, \omega$

$$\begin{array}{cc} \overrightarrow{OQ} & \overrightarrow{PQ} \\ 3 & 1 \end{array}$$

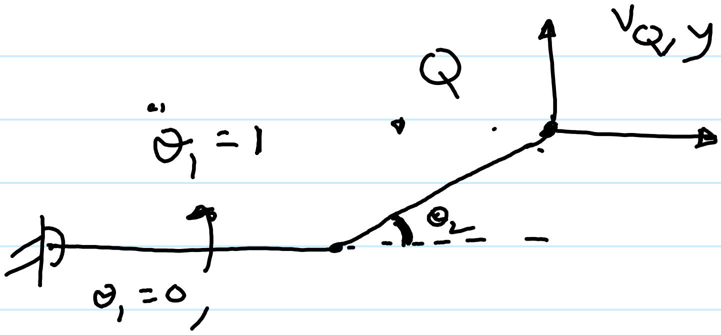
$$J_Q = \begin{bmatrix} -l\left[\sin(\theta_1+\theta_2)+\sin\theta_1\right] & -l\sin(\theta_1+\theta_2) \\ l\left[\cos(\theta_1+\theta_2)+\cos\theta_1\right] & l\cos(\theta_1+\theta_2) \end{bmatrix}$$

$$V_Q = J_Q\,w = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \dot\theta_1 \\ \dot\theta_2 \end{bmatrix}$$

$\dot\theta_1 = 1$ & $\dot\theta_2 = 0$

$$V_Q = \begin{bmatrix} J_{11} \\ J_{21} \end{bmatrix} = \begin{bmatrix} V_{Q,x} \\ V_{Q,y} \end{bmatrix}$$

$\theta_1 = 0$
$$\begin{cases} J_{11} = -l\sin\theta_2 \\ J_{21} = l\left[\cos\theta_2 + 1\right] \end{cases}$$

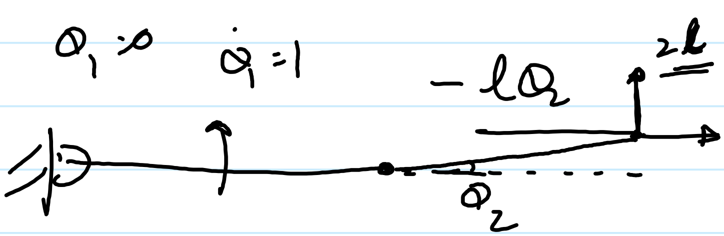


$V_{Q,y} = J_{21} = [1+\cos\theta_2)l$

$V_{Q,x} = J_{11}$

$= -l\sin\theta_2$

$\theta_2 \approx 0$



* $\dot{\theta}_1 = 1 \quad \dot{\theta}_2 = 0$

$$\left.\begin{aligned} J_{11} &= V_{Q,x} \\ J_{21} &= V_{Q,y} \end{aligned}\right\}$$

* $\dot{\theta}_1 = 0 \quad \dot{\theta}_2 = 1$

$$V_Q = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \cancelto{0}{\dot{\theta}_1} \\ \cancelto{1}{\dot{\theta}_2} \end{bmatrix}$$

$$\left\{\begin{aligned} V_{Q,x} &= J_{12} \\ V_{Q,y} &= J_{22} \end{aligned}\right.$$

Derivation by J.J. Craig

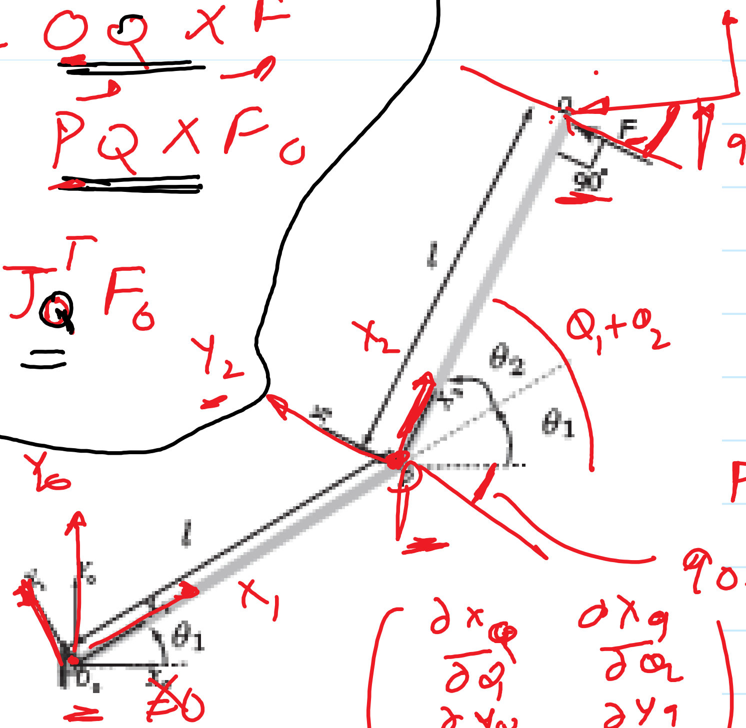
DH convention - kinematics.

(b)

(I) $\tau_1 = \vec{OQ} \times \vec{F}^0$

$\tau_2 = \vec{PQ} \times \vec{F}_0$

(II) $\begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix} = J_Q^T F_0$



(a)

$F^2 = 0\hat{X}_2 + F\hat{Y}_2$

$F^0 = \underline{\ ?\ }\,\hat{X}_0 + \underline{\ ?\ }\,\hat{Y}_0$

$$\begin{pmatrix} \frac{\partial x_Q}{\partial Q_1} & \frac{\partial x_Q}{\partial Q_2} \\ \frac{\partial y_Q}{\partial Q_1} & \frac{\partial y_Q}{\partial Q_2} \end{pmatrix}$$

2. Jacobian for static forces: Fig. 1 shows a two-link manipulator. Both link lengths are ℓ and the angles are θ_1 and θ_2 as shown. There is a force F that acts at right angles to the second link as shown. That is, the force acts along $Y2$.

   (a) Write an analytical expression for the force in fixed frame $O - X_0 - Y_0$ i.e., write the x- and y-component in the fixed frame.

   (b) Write an analytical expression for the torque at the two joints, $\tau = [\tau_1 \quad \tau_2]$.

(a)

(I) $F^0 = -F\cos(90-(\theta_1+\theta_2))\hat{X}_0 + F\sin(90-(\theta_1+\theta_2))\hat{Y}_0$

(II) $V^0 = R_1^0 V^1, \quad V^2 = R_1^0 R_2^1 V^2$

$F^0 = R_2^0 F^2 = R_1^0 R_2^1 F^2$

$$F^0 = \begin{bmatrix}\cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1\end{bmatrix}\begin{bmatrix}\cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2\end{bmatrix}\begin{bmatrix}0 \\ F\end{bmatrix}$$

Skew symmetric matrix

$$S(a) = \begin{bmatrix} 0 & -a_x & a_y \\ a_x & 0 & -a_z \\ -a_y & a_z & 0 \end{bmatrix} \Rightarrow S + S^T = 0$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \_\_\_\_\_\_$$

cross (a, b)

$$S(a)\, b = \begin{bmatrix} 0 & -a_x & a_y \\ a_x & 0 & -a_z \\ -a_y & a_z & 0 \end{bmatrix} \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \_\_\_\_\_\_$$

$$\left.\begin{aligned} \tau_1 &= \vec{OQ} \times \vec{F}^0 \\ \tau_2 &= \vec{PQ} \times \vec{F}^0 \end{aligned}\right\} \text{skew}$$

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = J^T F^0$$

3D

$$\begin{bmatrix} V \\ \omega \end{bmatrix} = \begin{bmatrix} J \end{bmatrix} \dot{\theta}$$

dof

$$\vec{\omega} \neq \dot{\theta}$$

$$V = \dot{\vec{\theta}} \times \vec{r} \quad \text{-2D}$$

$$V = J \dot{\theta}$$

$$\frac{\partial f}{\partial \theta}$$

$$\tau = \vec{r} \times \vec{F}$$

$$\tau = J^T F$$

$$\vec{a} \times \vec{b} \longrightarrow S(a)\, b$$

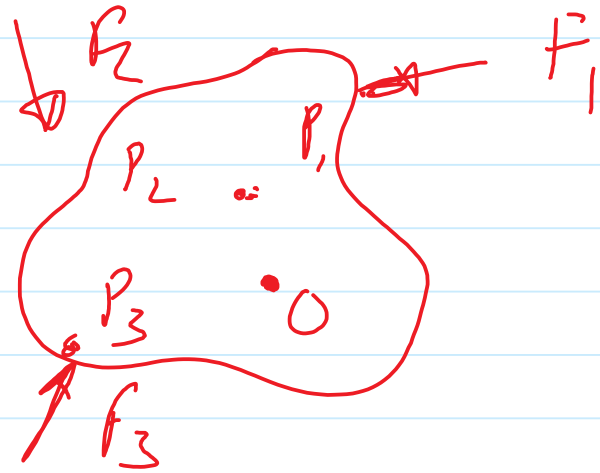
$$\vec{PQ} = \dots$$

$$p = (p_x, p_y)$$

$$q = (q_x, q_y)$$

$$\vec{PQ} = (q_x - p_x, \; q_y - p_y)$$

$$\tau = J^T F$$



$$V_{P_1} = J_{P_1} \dot{\theta}$$

$$\tau = J_{P_1}^T F_1 + J_{P_2}^T F_2 + J_{P_3}^T F_3$$

$$\vec{\tau}_O = \underline{\underline{\vec{OP_1} \times F_1}} + \underline{\underline{\vec{OP_2} \times F_2}} + \vec{OP_3} \times \vec{F_3}$$

cross

$$\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = J_{\varphi}^T \hat{F}^O$$